Complex Variables With Applications

Instructor: YONG YU

Department of Mathematics at CU

Sect.1. Complex Numbers and Some Basic Algebraic Manipulations

Different from real numbers, in the complex number theory, we have a number artificially introduced to make the algebraic equation $x^2 = -1$ solvable. This number is denoted by i, which is a pure imaginary number. In other words i is a number which satisfies $i^2 = -1$. With the notation i, we can define a complex number as follows

$$a + bi,$$

where a and b are all real numbers. Conventionally a is called real part of $a + bi$, while b is called the imaginary part of $a + bi$. We can also collect all complex numbers together and define

$$\mathbb{C} = \Big\{ a + bi : a \text{ and } b \text{ are real numbers} \Big\}.$$

The set $\mathbb{C}$ will be referred as complex field later on. In the complex field, all numbers with zero imaginary part are called real numbers, while all numbers with zero real part are called pure imaginary number.

Comparison between two complex numbers For two real numbers a and b, there are three relationships that may happen. They are $a < b$, $a = b$ or $a > b$. For complex numbers z_1 and z_2, we do not have $z_1 < z_2$ or $z_1 > z_2$ generally. But we can define $z_1 = z_2$.

Definition 1.1. *Suppose that $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$, where a_1, a_2, b_1, b_2 are four real numbers. Then we call $z_1 = z_2$ if and only if $a_1 = a_2$ and $b_1 = b_2$.*

Basically two complex numbers equal to each other if and only if their real parts and imaginary parts equal to each other, respectively.

Addition Two complex numbers can be added together.

Definition 1.2. *Suppose that $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$. Then we define $z_1 + z_2$ to be a complex number as follows:*

$$z_1 + z_2 = \big(a_1 + a_2\big) + \big(b_1 + b_2\big)i.$$

In terms of properties in real numbers, we also have

(i). Commutative Law: $z_1 + z_2 = z_2 + z_1$;

(ii). Associative Law: $z_1 + \big(z_2 + z_3\big) = \big(z_1 + z_2\big) + z_3$.

Here z_1, z_2 and z_3 are three arbitrary complex numbers. Also we have a particular number $0 + 0i$, simply denoted by 0, so that

$$z + 0 = z.$$

Any complex number added by 0 equals to itself. With the number 0, we can define summation inverse.

Definition 1.3. *Suppose that z is a complex number. Then the summation inverse, denoted by $-z$, of z is a complex number so that*

$$z + \big(-z\big) = 0.$$

Let $z = a + bi$ and let $-z = c + di$, where a, b, c and d are all real numbers. Then by the Definition 1.3 and the definition of addition, we must have

$$z + \big(-z\big) = \big(a + c\big) + \big(b + d\big)i = 0.$$

Using Definition 1.1 then yields $a + c = 0$ and $b + d = 0$. That is $c = -a$ and $d = -b$. In other words if $z = a + bi$, then its addition inverse is read as $-a + \big(-b\big)i$. Formally we take -1 in front as a common factor. The addition inverse is then read as $-\big(a + bi\big) = -z$. This is the origin of the notation $-z$. With the concept of addition inverse, subtraction can also be introduced

Definition 1.4 (Subtraction of two complex numbers)**.** *Suppose that* $z_1 = a_1 + b_1 i$ *and* $z_2 = a_2 + b_2 i$. *Then we define*

$$z_1 - z_2 := z_1 + \big(-z_2\big) = \big(a_1 - a_2\big) + \big(b_1 - b_2\big)i.$$

Product Two complex numbers can be multiplied. Formally we can apply the distributive law that we have learned before. If $z_1 = a + bi$ and $z_2 = c + di$, then formally if the distributive law and commutative law hold for complex numbers, then it should satisfy

$$z_1 z_2 = \big(a + bi\big)\big(c + di\big) = ac + adi + bci + bdi^2.$$

The last term in the above has i^2. But as we know i is introduced so that $i^2 = -1$. Then we can rewrite the above equality as follows

$$z_1 z_2 = \big(a + bi\big)\big(c + di\big) = ac + adi + bci - bd = \big(ac - bd\big) + \big(ad + bc\big)i.$$

If distributive law and commutative law for real numbers still hold for complex numbers, then the most-right-hand side above is the only number that we can have. Motivated by this consideration, we define

Definition 1.5.

$$z_1 z_2 = \big(ac - bd\big) + \big(ad + bc\big)i,$$

if $z_1 = a + bi$ *and* $z_2 = c + di$.

With this definition, we can easily show

$$\big(z_1 + z_2\big)^n = \sum_{k=0}^{n} C_n^k \, z_1^k \, z_2^{n-k}, \qquad z_1 \text{ and } z_2 \text{ are complex numbers, } n \text{ is a natural number.}$$

This formulae is the so-called binomial formulae. One can also show
(i). Commutative Law: $z_1 z_2 = z_2 z_1$;
(ii). Associative Law: $\big(z_1 z_2\big)z_3 = z_1\big(z_2 z_3\big)$;
(iii). Distributive Law: $z_1\big(z_2 + z_3\big) = z_1 z_2 + z_1 z_3$.
Here z_1, z_2 and z_3 are three arbitrary complex numbers. Also we have a particular number $1 + 0i$, simply denoted by 1, so that any complex number multiplied by 1 equals to itself. With the number 1, we can define product inverse.

Definition 1.6. *Suppose that* z *is a complex number. Then the product inverse, denoted by* $\frac{1}{z}$, *of* z *is a complex number so that*

$$z\,\frac{1}{z} = 1.$$

$\frac{1}{z}$ *is also denoted by* z^{-1} *sometimes in the future.*

How to compute z^{-1} ? Suppose $z = a + bi$ and $z^{-1} = c + di$. By Definition 1.6, it must hold

$$zz^{-1} = \big(ac - bd\big) + \big(ad + bc\big)i = 1.$$

Compare real parts and imaginary parts. c and d are solutions to the following linear equation:

$$\begin{cases} ac - bd = 1; \\ bc + ad = 0. \end{cases} \tag{1.1}$$

This system has a unique solution if and only if

$$\det \begin{pmatrix} a, & -b \\ b, & a \end{pmatrix} = a^2 + b^2 \neq 0.$$

Therefore we know z^{-1} cannot be defined if $z = 0$. Moreover if $z = a + bi \neq 0$, then (1.1) yields

$$c = \frac{a}{a^2 + b^2}, \qquad d = -\frac{b}{a^2 + b^2}.$$

Equivalently

$$z^{-1} = \frac{1}{z} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}\, i, \qquad \text{if } z = a + bi.$$

With the product inverse, we can also define division. Given z_1 and z_2 where $z_2 \neq 0$, we let

$$\frac{z_1}{z_2} = z_1 \frac{1}{z_2}.$$

So far we have talked about some algebraic manipulations of complex numbers. Now we take a look at one of its applications.

Euler's Formulae For real numbers, we have definition of e^x. Can we define e^z when z is a complex number ? For the real case, exponential function satisfies

$$e^{x+y} = e^x e^y. \tag{1.2}$$

We hope this property still holds for complex numbers. Therefore if $z = a + bi$ and (1.2) holds for complex numbers, then

$$e^z = e^{a+bi} = e^a e^{bi} \tag{1.3}$$

Here a and b are real numbers. Notice that e^a is now well-defined. But e^{bi} still has no definition so far. To define e^{bi} where b is a real number, we need to recall the second property of the exponential function in the real case. In fact e^x when x is real admits a Taylor expansion. That is for any x,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

If the above expansion holds for complex number, particularly for the pure imaginary number, then we must have

$$e^{bi} = \sum_{n=0}^{\infty} \frac{b^n i^n}{n!}.$$

Since $i^{4n} = 1$, $i^{4n+1} = i$, $i^{4n+2} = -1$, $i^{4n+3} = -i$, the last equality formally can be reduced to

$$\begin{aligned} e^{bi} = \sum_{n=0}^{\infty} \frac{b^n i^n}{n!} &= \sum_{k=0}^{\infty} \frac{b^{4k} i^{4k}}{(4k)!} + \sum_{k=0}^{\infty} \frac{b^{4k+1} i^{4k+1}}{(4k+1)!} + \sum_{k=0}^{\infty} \frac{b^{4k+2} i^{4k+2}}{(4k+2)!} + \sum_{k=0}^{\infty} \frac{b^{4k+3} i^{4k+3}}{(4k+3)!} \\ &= \sum_{k=0}^{\infty} \frac{b^{4k}}{(4k)!} + i \sum_{k=0}^{\infty} \frac{b^{4k+1}}{(4k+1)!} - \sum_{k=0}^{\infty} \frac{b^{4k+2}}{(4k+2)!} - i \sum_{k=0}^{\infty} \frac{b^{4k+3}}{(4k+3)!}. \end{aligned}$$

We now combine real parts and imaginary parts above together. It then follows

$$e^{bi} = \left(\sum_{k=0}^{\infty} \frac{b^{4k}}{(4k)!} - \sum_{k=0}^{\infty} \frac{b^{4k+2}}{(4k+2)!} \right) + i \left(\sum_{k=0}^{\infty} \frac{b^{4k+1}}{(4k+1)!} - \sum_{k=0}^{\infty} \frac{b^{4k+3}}{(4k+3)!} \right). \tag{1.4}$$

For the real part above, it holds

$$\sum_{k=0}^{\infty} \frac{b^{4k}}{(4k)!} - \sum_{k=0}^{\infty} \frac{b^{4k+2}}{(4k+2)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} b^{2n}$$

The right-hand side is the Taylor expansion of $\cos b$. Then we get

$$\sum_{k=0}^{\infty} \frac{b^{4k}}{(4k)!} - \sum_{k=0}^{\infty} \frac{b^{4k+2}}{(4k+2)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} b^{2n} = \cos b. \tag{1.5}$$

Similarly

$$\sum_{k=0}^{\infty} \frac{b^{4k+1}}{(4k+1)!} - \sum_{k=0}^{\infty} \frac{b^{4k+3}}{(4k+3)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} b^{2n+1} = \sin b. \tag{1.6}$$

Applying (1.5)-(1.6) to (1.4) yields

$$e^{bi} = \cos b + i \sin b. \tag{1.7}$$

This formulae is the famous Euler's formulae. One should pay attention that, the above calculations are formally true. Formally means we assume (1.3) and the Taylor expansion of e^z holds in the complex scenario. Now we use (1.7) and (1.3) to define the complex exponential function. That is

Definition 1.7. *For any real numbers a and b, we let*

$$e^z = e^{a+b\,i} := e^a e^{b\,i} := e^a \left(\cos b + i \sin b \right).$$

With the Definition 1.7, we easily have

Proposition 1.8.

$$e^{z_1+z_2} = e^{z_1} e^{z_2}, \qquad \textit{for any two complex numbers } z_1 \textit{ and } z_2.$$

Moreover we also have the complex version of Taylor expansion of e^z. That is

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

We won't prove this expansion here. It will be rigorously shown later on, based on the Definition 1.7.

Sect.2. Geometric Representation of Complex Number Field.

Now we turn to the geometric representation of a complex number. Basically a complex number can be determined if we have its real and imaginary parts. In other words a complex number can be identified with a point in $\mathbb{R}^2$ space. In fact we construct the following correspondence. For a complex number, denoted by $a+bi$, we relate it to the point (a,b) in $\mathbb{R}^2$. The first coordinate of (a,b) is the real part of $a+bi$, while the second coordinate of (a,b) is the imaginary part of $a+bi$. By this way we obtain a one-one correspondence between $\mathbb{C}$ and $\mathbb{R}^2$. We can visualize a complex number geometrically. More than this, all the algebraic manipulations introduced in Sect.1 can also be explained in a geometric way.

Addition Given z_1 and z_2 two complex numbers, their corresponding points in $\mathbb{R}^2$ are also denoted by z_1 and z_2. Then by using 0, z_1 and z_2, we can construct a parallelogram in $\mathbb{R}^2$. $\overline{0z_1}$ and $\overline{0z_2}$ are two edges of the parallelogram. Clearly the fourth vertex in the parallelogram (different from 0, z_1 and z_2) corresponds to the complex number z_1+z_2;

Subtraction Given z_1 and z_2 two complex numbers, their corresponding points in $\mathbb{R}^2$ are also denoted by z_1 and z_2. $-z_2$ is the point symmetric to z_2 with respect to the origin. Using 0, $-z_2$ and z_1, we can construct a parallelogram. Then the fourth vertex on this parallelogram (different from 0, $-z_2$, z_1) denotes the complex number z_1-z_2. Pay attention z_1-z_2 denotes the vector **starting from z_2 and ending at z_1.**

As we know, in $\mathbb{R}^2$ space, besides the Euclidean coordinate, we can also represent a point in $\mathbb{R}^2$ by polar coordinate. Suppose that (ρ,θ) is the polar coordinate of the point (x,y) in $\mathbb{R}^2$. Then the Euclidean coordinates for point (x,y), (represented in terms of (ρ,θ)), can be computed by

$$(x,y)=\Big(\rho\cos\theta,\rho\sin\theta\Big).$$

Using the correspondence between $\mathbb{C}$ and $\mathbb{R}^2$, we know that

$$z:=x+yi=\rho\cos\theta+i\rho\sin\theta=\rho\Big(\cos\theta+i\sin\theta\Big).$$

Now we apply the Euler's formulae to get

$$\rho\cos\theta+i\rho\sin\theta=\rho\Big(\cos\theta+i\sin\theta\Big)=\rho e^{i\theta}.$$

Therefore the above two equalities yield

$$z=x+iy=\rho e^{i\theta}.$$

The last term above is called polar representation of the complex number $x+iy$. In the polar coordinate, ρ is the distance between (x,y) and the origin. It is uniquely determined and equals to

$$\rho=\sqrt{x^2+y^2}.$$

In the theory of complex numbers, for a given complex number $z=x+yi$, we denote by $|z|$ the quantity ρ. $|z|$ is referred as modulus of z in the following course. θ in the polar representation of a complex number $z=x+yi$ is the angle between the vector (x,y) and the positive direction of the x-axis. Obviously if we don't restrict the range of θ, the angular variable for a vector (x,y) takes multiple values. In fact cos and sin are periodic functions with period 2π. If (ρ,θ) is a polar coordinate of a point (x,y), then $(\rho,\theta+2k\pi)$ represent the same point (x,y). Here k is any integer number. In complex theory, θ in the polar coordinate of (x,y) is called argument of the complex number $x+iy$.

Remark 1.9. *If we restrict θ to be a number in $(-\pi,\pi]$, then the argument for a complex number can be uniquely determined. But generally argument corresponding to a complex number is multiple. Two arguments for an associated complex number are different from each other by $2k\pi$, where k is an integer. In the future, we call the argument of z in the interval $(-\pi,\pi]$ the principal argument and denote it by* $\mathrm{Arg}(z)$. *Given a complex number $z\neq 0$,* $\arg(z)$ *is the notation for the following set*

$$\arg(z):=\Big\{\mathrm{Arg}(z)+2k\pi : k \textit{ is an integer}\Big\}.$$

The modulus of a complex number z satisfies the following triangle inequality

Proposition 1.10. *For any complex numbers z_1 and z_2, it holds*

$$|z_1 + z_2| \leqslant |z_1| + |z_2|.$$

Proof. Given z_1 and z_2, we can construct a triangle with vertices 0, z_1 and $z_1 + z_2$. The distance between 0 and $z_1 + z_2$ is bounded by the summation of the lengths of the remaining two edges in the triangle. The proof then follows easily. □

Using inductive arguments, we also have

Proposition 1.11. *Suppose that z_1 , ..., z_n are n complex numbers, then it holds*

$$|z_1 + ... + z_n| \leqslant |z_1| + ... + |z_n|.$$

In the following, let us take a look at geometric meaning of multiplication. Suppose $z = \rho e^{i\theta}$ and $z_0 = \rho_0 e^{i\theta_0}$. Here (ρ, θ) and (ρ_0, θ_0) are polar coordinates of z and z_0, respectively. Then by Proposition 1.8, it holds

$$zz_0 = \rho\rho_0 e^{i\left(\theta+\theta_0\right)}. \tag{1.8}$$

Notice that the modulus of zz_0 equals $\rho\rho_0$. This shows

$$\left|zz_0\right| = \rho\rho_0 = |z||z_0|.$$

The argument of zz_0 equals to

$$\left\{\theta + \theta_0 + 2k\pi : k \text{ is an integer }\right\}. \tag{1.9}$$

Remark 1.12. *Usually we have*

$$\arg\left(zz_0\right) = \arg(z) + \arg(z_0).$$

This notation is meaningful in the sense of set addition. More precisely the left-hand side of the above equality is given by (1.9). The right-hand side of the above equality is understood as

$$\left\{x + y : x \in \arg(z), y \in \arg(z_0)\right\}.$$

But generally the equality

$$\mathrm{Arg}\left(zz_0\right) = \mathrm{Arg}(z) + \mathrm{Arg}(z_0)$$

is false. For example, $z = -1$ and $z_0 = i$. Clearly $\mathrm{Arg}\left(zz_0\right) = \mathrm{Arg}\left(-i\right) = -\frac{\pi}{2}$. However $\mathrm{Arg}(-1) = \pi$, while $\mathrm{Arg}(i) = \frac{\pi}{2}$. It then follows $\mathrm{Arg}(-1) + \mathrm{Arg}(i) = \frac{3\pi}{2}$.

Now we go back to (1.8) and understand more clearly the geometric meaning of complex product. If $\rho_0 = 1$, then the modulus of zz_0 equals to the modulus of z. If $\rho_0 > 1$, then the modulus of zz_0 is longer than the modulus of z. It is stretched. If $0 < \rho_0 < 1$, then the modulus of zz_0 is shortened. As for the argument of zz_0, if $\theta_0 = 0$, then the argument keeps to be θ. The direction of zz_0 and z are the same. If $\theta_0 > 0$, then the argument of zz_0 equals to $\theta + \theta_0$. In this case we need rotate the direction of z counterclockwisely by θ_0 so that the rotated vector can have the same direction as zz_0. If $\theta_0 < 0$, then the argument of zz_0 equals to $\theta - \left(-\theta_0\right)$. In this case we need rotate the direction of z clockwisely by $|\theta_0| = -\theta_0$ so that the rotated vector can have the same direction as zz_0. In summary, if we multiply a complex number z by a positive number, then it corresponds to stretch or compress the vector z. But meanwhile the direction is fixed. If we multiply a complex z by a complex number $e^{i\theta}$, then it corresponds to rotate z by the angle $|\theta|$ counter-clockwisely (if $\theta > 0$) or clockwisely (if $\theta < 0$). Meanwhile the length is fixed. Multiplying z by a general complex number z_0 correspond to a composed operation of both stretching and rotation.

With the properties introduced above, we consider

Example Using triangle inequality to estimate $3 + z + z^2$ for all z with modulus 2.

Solution. By triangle inequality, it holds

$$\left|3 + z + z^2\right| \leqslant 3 + |z| + |z^2|.$$

Since $|z^2| = |z|^2$ and $|z| = 2$, the above estimate is reduced to

$$\left|3 + z + z^2\right| \leqslant 3 + |z| + |z^2| = 3 + |z| + |z|^2 = 9.$$

Sect.3 Some Basic Geometric Objects Represented In Complex Theory.

Using the quantities in Sect.2, we can represent some geometric objects in complex theory.
Example 1. A circle with center z_0 and radius r_0 is given by $\left\{z \in \mathbb{C} : |z - z_0| = r_0\right\}$.
Example 2. Interior part of the circle given in Example 1 is the set $\left\{z \in \mathbb{C} : |z - z_0| < r_0\right\}$.
Example 3. Exterior part of the circle given in Example 1 is the set $\left\{z \in \mathbb{C} : |z - z_0| > r_0\right\}$.
Example 4. Ellipsis with foci z_1 and z_2 is given by $\left\{z \in \mathbb{C} : |z - z_1| + |z - z_2| = d\right\}$. Here d is the length of the long axis.
Example 5. Lines in $\mathbb{C}$. Given z_1 and z_2 two complex numbers in $\mathbb{C}$, they decide a straight line l so that l passes across z_1 and z_2. For all points on l, denoted by z, the direction from z_1 to z_2 and the direction from z_1 to z are either the same or differ by π. Therefore by polar coordinates, if $z_2 - z_1 = \rho e^{i\theta}$, then it must hold

$$z - z_1 = re^{i\theta} \qquad \text{or} \qquad z - z_1 = re^{i\left(\theta+\pi\right)}.$$

Here ρ and r are modulus of $z_2 - z_1$ and $z - z_1$, respectively. Therefore we have

$$\text{either} \quad \frac{z - z_1}{z_2 - z_1} = \frac{r}{\rho} \qquad \text{or} \qquad \frac{z - z_1}{z_2 - z_1} = -\frac{r}{\rho}.$$

In either case, the argument of $\left(z - z_1\right)/\left(z_2 - z_1\right)$ is 0, provided that z lies on the line l. The converse is also true. So in the complex theory, line l determined by z_1 and z_2 can be represented by

$$\left\{z \in \mathbb{C} : \operatorname{Im}\left(\frac{z - z_1}{z_2 - z_1}\right) = 0\right\}. \tag{1.10}$$

Example 6. In Example 1, we have given an analytic way to represent a circle. In complex theory, we have a second way to represent a circle. As we know a circle can be uniquely determined if we are given three points which are not on the same line. Suppose that the circle C_1 is the circle passing across z_1, z_2 and z_3. Here z_1, z_2 and z_3 are three points on C_1 and they are clockwisely distributed. Suppose that z is another point on C_1. Without loss of generality we assume z lies on C_1 so that z_1, z_2, z_3 and z are clockwisely distributed. Other cases can be similarly considered. Then by fundamental geometry, it holds

$$\angle z_1 z_3 z_2 = \angle z_1 z z_2.$$

The reason is that these two angles correspond to the same arc on the circle C_1. Notice that we can rotate the vector $z_3 - z_2$ counterclockwisely by the angle $\angle z_1 z_3 z_2$, the resulted vector must have the same direction as $z_3 - z_1$. Therefore we have

$$z_3 - z_1 = \lambda_1\left(z_3 - z_2\right)e^{i\angle z_1 z_3 z_2}, \qquad \text{for some } \lambda_1 > 0.$$

Similarly we have

$$z - z_1 = \lambda_2 (z - z_2) e^{i\angle z_1 z z_2}, \qquad \text{for some } \lambda_2 > 0.$$

Here λ_1 and λ_2 are positive real numbers. Since $\angle z_1 z_3 z_2 = \angle z_1 z z_2$, the last two equalities yield

$$\left(\frac{z - z_1}{z - z_2}\right) \Big/ \left(\frac{z_3 - z_1}{z_3 - z_2}\right) = \frac{\lambda_2}{\lambda_1}.$$

This furthermore implies

$$\operatorname{Im}\left[\left(\frac{z - z_1}{z - z_2}\right) \Big/ \left(\frac{z_3 - z_1}{z_3 - z_2}\right)\right] = 0.$$

One can apply similar arguments above for the other possible positions of z on C_1. The last equality always hold once z is on C_1. Therefore we conclude that

$$C_1 = \left\{ z \in \mathbb{C} : \operatorname{Im}\left[\left(\frac{z - z_1}{z - z_2}\right) \Big/ \left(\frac{z_3 - z_1}{z_3 - z_2}\right)\right] = 0 \right\}. \tag{1.11}$$

After Examples 5 and 6, we take a look at some more examples on their application.

Example 7. Find all points which satisfy

$$\operatorname{Im}\left(\frac{z + 1 - 3i}{4 - i}\right) = 0.$$

The condition given in this example is quite similar to (1.10). It is a particular case of (1.10) when we have

$$-z_1 = 1 - 3i, \qquad z_2 - z_1 = 4 - i.$$

Equivalently it holds $z_1 = -1 + 3i$, $z_2 = 3 + 2i$. By the discussion in Example 5, the points in this example represent a line passing across $-1 + 3i$ and $3 + 2i$.

Example 8. Find all points which satisfy

$$\operatorname{Im}\left(\frac{1}{z}\right) = 1.$$

Notice that

$$\operatorname{Im}\left(\frac{1}{z}\right) = 1 = \operatorname{Im}(i).$$

Therefore

$$0 = \operatorname{Im}\left(\frac{1}{z} - i\right) = \operatorname{Im}\left(\frac{1 - iz}{z}\right) = \operatorname{Im}\left(\frac{z + i}{z} \cdot (-i)\right).$$

Compare with (1.11), we have in this example

$$-z_1 = i, \qquad z_2 = 0, \qquad \frac{z_3 - z_2}{z_3 - z_1} = -i.$$

Equivalently it holds $z_1 = -i$, $z_2 = 0$, $z_3 = \frac{1}{2} - \frac{i}{2}$. It represents a circle passing across these three points. Analytically all points in this example satisfy

$$\left| z + \frac{i}{2} \right| = \frac{1}{2}.$$

Example 9. Side of a line. Given different z_1 and z_2 in $\mathbb{C}$, we can determine a line. There are two directions if a line is given. One direction is from z_1 to z_2, while another direction is from z_2 to z_1. The concept of side is related to the direction that we are using. If we fix a direction by starting from z_1 to z_2, then all points on the left form the left-side of the line l, while all points on the right form the right-side of the line l. Pay attention: Left and Right sides depend on the direction that we are using. Suppose the direction is given by starting from z_1 to z_2. Then for an arbitrary point z on the left-side, we can rotate $z_2 - z_1$ counter-clockwisely by an angle θ_0 to the direction given by $z - z_1$. Since z is on the left-side, this θ_0 can be in the interval $(0, \pi)$. In other words,

$$z - z_1 = \lambda(z_2 - z_1)e^{i\theta_0}, \qquad \text{for some } \lambda > 0 \text{ and } \theta_0 \in (0, \pi).$$

From the above equality we have

$$\mathrm{Im}\left(\frac{z - z_1}{z_2 - z_1}\right) = \lambda \sin\theta_0 > 0.$$

Similarly if z is on the right-side of l with the direction given by pointing from z_1 to z_2, then it holds

$$\mathrm{Im}\left(\frac{z - z_1}{z_2 - z_1}\right) < 0.$$

The above arguments and (1.10) implies that given z_1 and z_2, all points satisfy (1.10) must lie on the line across z_1 and z_2. If

$$\mathrm{Im}\left(\frac{z - z_1}{z_2 - z_1}\right) > 0,$$

then z lies on the left-side of l. The direction is from z_1 to z_2. If

$$\mathrm{Im}\left(\frac{z - z_1}{z_2 - z_1}\right) < 0,$$

then z lies on the right-side of l. The direction is from z_1 to z_2.

Example 10. Find all points satisfying

$$\mathrm{Im}\left(\frac{z + 1 - 3i}{4 - i}\right) > 0. \tag{1.12}$$

By example 7, $z_1 = -1 + 3i$, $z_2 = 3 + 2i$. By Example 9, z satisfying (1.12) must be on the left-side. The left-side is determined by the direction from z_1 to z_2.

Example 11. Symmetric point with respect to x-axis. In complex theory, given a complex number $z = x + iy$, we have an operator to find its symmetric point with respect to x-axis. In fact the symmetric point of (x, y) with respect to x-axis is $(x, -y)$. This symmetric point corresponds to the number $x - iy$. In the future, we denote by $\overline{z} = x - iy$ the symmetric point and call it conjugate number of z. The following formulaes can be easily shown

$$\mathrm{Re}(z) = \frac{z + \overline{z}}{2}, \quad \mathrm{Im}(z) = \frac{z - \overline{z}}{2i}, \quad \overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}, \quad |z| = |\overline{z}|.$$

Example 12. Computation of roots. Given $z = \rho e^{i\theta}$, we can easily calculate $z^n = \rho^n e^{in\theta}$. Conversely if we are given $a = \rho_0 e^{i\theta_0} \neq 0$, we can also find z such that $z^n = a$. Here n is a natural number. Indeed suppose that $z = \rho e^{i\theta}$, then $z^n = a$ can be equivalently written as

$$\rho^n e^{in\theta} = \rho_0 e^{i\theta_0}$$

It then follows

$$\rho = \rho_0^{1/n}, \qquad e^{i\left(n\theta - \theta_0\right)} = 1.$$

ρ is uniquely determined. But since cos and sin functions are periodic function, the second equality above can only imply

$$n\theta - \theta_0 = 2k\pi, \qquad k \text{ is an integer.}$$

Therefore θ is not uniquely determined. All z with $\rho = \rho_0^{1/n}$ and θ given by

$$\frac{\theta_0}{n} + \frac{2k\pi}{n}$$

will satisfy the equation $z^n = a$. Such z is called n-th root of a. Notice that we can only have n different roots for a given non-zero complex number a. For example $(-16)^{1/4}$. In this case, $\rho_0 = 16$, $\theta_0 = \pi$, $n = 4$. Therefore the fourth root of -16 are

$$2e^{i\pi/4}, \quad 2e^{i3\pi/4}, \quad 2e^{i5\pi/4}, \quad 2e^{i7\pi/4}.$$

We remark here that if $a = 0$, then 0 is the only solution for $z^n = 0$. Therefore the n-th roots of 0 are all zero.

Sect. 4. Functions on Subset of Complex Plane.

Starting from this section, we study functions defined on complex numbers. Basically functions are different rules which send points in some subset to their corresponding complex values. Formally a function can be written as

$$f : S \longrightarrow \mathbb{C}. \tag{1.13}$$

In (1.13), f is called function name. S is a subset of $\mathbb{C}$ on which f is defined. The last $\mathbb{C}$ in (1.13) means f takes complex values. Since a complex number can be represented by $a + bi$ with a and b complex numbers, then by the above description in (1.13), we can represent f by $f_1 + f_2 i$, where f_1 and f_2 are two real valued functions defined on S. Notice that to define a function, we need

(1). A subset S of $\mathbb{C}$ on which f can be defined;

(2). For each given $z \in S$, there is only one number, denoted by $f(z)$, corresponding to the number z under the rule given by the function f.

Domain of a Function. S in (1.13) is called domain of a function f. S could be the whole set $\mathbb{C}$. But in many cases, S is only part of complex plane $\mathbb{C}$. Intuitively you can imagine $\mathbb{C}$ as a whole piece of paper. You can use a pencil to draw a closed loop, denoted by γ, on the paper and then cut along the closed loop γ. By this way, we can obtain a part of the paper, denoted by Ω, which contains the interior of the closed loop γ. Keep this part of paper and then draw more closed loops on this part. These loops are denoted by γ_j with $j = 1, ..., n$. Each γ_j should has no intersection with others. Then you can keep cutting along γ_j $(j = 1, ..., n)$. Finally you will see that what we are left on Ω will form a part of the paper with finitely many holes. Mathematically we will call this remaining part of $\mathbb{C}$ a multiple connected domain with exterior boundary γ and interior boundaries γ_j $(j = 1, ..., n)$. Moreover you can see that Ω obtained before only has exterior boundary γ without γ_j $(j = 1, ..., n)$. Such Ω will be called simply connected domain with boundary γ.

To be more precise, let us take a look at Example 2 in Sect. 3. Given $z_0 \in \mathbb{C}$ and $r_0 > 0$, the interior part of the circle with center z_0 and radius r_0 is read as

$$D(z_0; r_0) := \Big\{ z \in \mathbb{C} : |z - z_0| < r_0 \Big\}.$$

$D(z_0; r_0)$ is called open disk in $\mathbb{C}$. Obviously it is enclosed by a close loop which is actually the circle

$$\text{Cir}(z_0; r_0) := \Big\{ z \in \mathbb{C} : |z - z_0| = r_0 \Big\}.$$

Assuming that $z_0 = x_0 + iy_0$ and using $\theta \in (0, 2\pi]$ as a parameter, then we can represent $\mathrm{Cir}(z_0; r_0)$ by the following parametrization:

$$\Big\{z = \big(x_0 + r_0 \cos\theta\big) + \big(y_0 + r_0 \sin\theta\big)i : \theta \in (0, 2\pi]\Big\}. \tag{1.14}$$

As θ runs in $(0, 2\pi]$, $\big(x_0 + r_0 \cos\theta\big) + \big(y_0 + r_0 \sin\theta\big)i$ sweeps out all the points on $\mathrm{Cir}(z_0; r_0)$. Meanwhile there is no two different angles in $(0, 2\pi]$ which correspond to the same point on $\mathrm{Cir}(z_0; r_0)$. Using $\mathrm{Cir}(z_0; r_0)$, we can enclose the open disk $D(z_0; r_0)$. Roughly speaking, $\mathrm{Cir}(z_0; r_0)$ helps us cut out a region in $\mathbb{C}$. Generally we can use closed loop with any shape to cut a region out of $\mathbb{C}$. This motivates us to generalize a little bit the parametrization in (1.14). Notice that the parametrization in (1.14) has some properties.

(1). The parametrization is differentiable with respect to the variable θ. That is both $x_0 + r_0 \cos\theta$ and $y_0 + r_0 \cos\theta$ are differentiable functions with respect to the variable θ;

(2). For any different $\theta \in (0, 2\pi]$, $\big(x_0 + r_0 \cos\theta\big) + \big(y_0 + r_0 \sin\theta\big)i$ corresponds to different points on $\mathrm{Cir}(z_0; r_0)$;

(3). If we allow $\theta = 0$, then at the two end-points of $[0, 2\pi]$, the parametrization $z = \big(x_0 + r_0 \cos\theta\big) + \big(y_0 + r_0 \sin\theta\big)i$ takes the same value. Intuitively, the curve given by (1.14) are connected at the two end-points.

Based on the three points above, we define

Definition 1.13. *γ is called a differentiable closed loop in $\mathbb{C}$ if γ can be parameterized by*

$$\Big\{z = f_1(s) + f_2(s)i : s \in (a, b]\Big\}$$

with f_1 and f_2 satisfying

(1). For all $s \in (a, b]$, f_1 and f_2 are two differentiable real-valued functions of the variable s;

(2). For any s_1, s_2 in $(a, b]$ with $s_1 \neq s_2$, it holds $f_1(s_1) + f_2(s_1)i \neq f_1(s_2) + f_2(s_2)i$;

(3). The following two limits hold

$$\lim_{s \to a+} f_1(s) = f_1(b), \qquad \lim_{s \to a+} f_2(s) = f_2(b).$$

Notice that (3) in Definition 1.13 is used to connect two end-points of γ at a same location. Similarly to the circle case, for any given γ a closed differentiable loop, γ also encloses a bounded region in $\mathbb{C}$, i.e. the interior part of γ. Such region will be referred as **Simply Connected Region with Differentiable Boundary γ.** Moreover if we denote this interior region by Ω, then the union $\Omega \cup \gamma$ is called closure of Ω and is denoted by $\overline{\Omega}$.

We can also cut finitely many sub-regions from a given simply connected region with boundary γ. More precisely let Ω be a simply connected region with boundary γ. Ω_1, Ω_2, ..., Ω_n are n subsets of Ω. For each $j = 1, ..., n$, Ω_j is also a simply connected region with a boundary γ_j. If it holds

$$(i).\ \overline{\Omega}_j \bigcap \overline{\Omega}_k = \emptyset, \quad \text{for } j \neq k; \qquad (ii).\ \bigcup_{j=1}^{n} \overline{\Omega}_j \subset \Omega,$$

then we can subtract the union of $\overline{\Omega}_j$ from Ω and obtain

$$\Omega \setminus \bigcup_{j=1}^{n} \overline{\Omega}_j.$$

Clearly the boundary of the last set contains multiple portions. Besides γ, the boundary of Ω, γ_j $(j = 1, ..., n)$ are also boundaries of $\Omega \setminus \bigcup_{j=1}^{n} \overline{\Omega}_j$. In the following course the set $\Omega \setminus \bigcup_{j=1}^{n} \overline{\Omega}_j$ will be called **multiple connected domain with exterior boundary γ and interior boundary γ_1, ..., γ_n.** A simple example of multiple connected domain is the annulus

$$A(z_0; r_1, r_2) := \Big\{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\Big\}. \tag{1.15}$$

The circle $\mathrm{Cir}(z_0; r_1)$ is the interior boundary of $A(z_0; r_1, r_2)$, while $\mathrm{Cir}(z_0; r_2)$ is the exterior boundary of $A(z_0; r_1, r_2)$.

In the above discussions, boundary curves are all differentiable. But in applications, boundary curves might also admit some corners. For example, rectangles. The boundary of a rectangle is differentiable for almost all points except four vertices. In this course, we also allow boundary of a simply connected region (or multiple connected region) to be piecewisely differentiable. This equivalently tells us that parametrization of boundary curves are piecewisely differentiable functions.

Besides simply connected and multiple connected domain discussed above, we also need the concept of open set in Ω.

Definition 1.14 (Open Set)**.** *Ω is a subset of $\mathbb{C}$. It is called an open set if for any $z_0 \in \Omega$, we can always find a tiny $r_0 > 0$ so that $D(z_0; r_0) \subset \Omega$.*

With the definition of open set, we define

Definition 1.15 (Closed set)**.** *Suppose that T is a subset of $\mathbb{C}$. It is called a closed set if its complement set $\mathbb{C}\backslash T$ is an open set.*

Examples of Functions. Now we take a look at some examples.

Example 1. $f(z) = z^2$. This is a quadratic equation. Suppose $z = x + iy$, then $f(z) = \big(x^2 - y^2\big) + 2xyi$. Clearly the real part of f is $x^2 - y^2$ and the imaginary part is read as $2xy$. Both of these two functions can be defined on the whole set $\mathbb{C}$. Therefore we know that the domain of z^2 is $\mathbb{C}$;

Example 2. $f(z) = |z|^2$. Suppose $z = x + iy$, then $f(z) = x^2 + y^2$. Its real part is $x^2 + y^2$, while its imaginary part is 0. Domain is also $\mathbb{C}$.

Example 3. Function in Example 1 can be generalized a little bit. Given a natural number n and $n+1$ complex numbers a_0, a_1, ..., a_n, we denote by $P_n(z)$ the function $a_0 + ... + a_n z^n$. This function is called polynomial of order n, provided that the coefficient $a_n \neq 0$. The number n is also called the order of the polynomial P_n. The domain of P_n is also $\mathbb{C}$;

Example 4. Given two polynomials, denoted by $P(z)$ and $Q(z)$, we can compute $R(z) = P(z)/Q(z)$. $R(z)$ is called rational functions. In this case, $R(z)$ cannot generally be evaluated on all points in $\mathbb{C}$. Since we have a denominator $Q(z)$, the function $R(z)$ has no definition generally on points at where Q equals to 0. For example $R(z) = (z + 3)/(z + 1)$. The denominator equals to 0 when $z = -1$. The domain of R in this case has no definition at $z = -1$. Therefore the domain of R in this case is $\mathbb{C}\backslash\{-1\}$.

Example 5. Consider $z^{1/2}$, the square root of a complex number z. From Example 12 in Sect. 3, we know that square root of a complex number z is not unique. There are two numbers corresponding to the square root of z. Therefore by the definition of a function, $z^{1/2}$ is not a function in general since it will be confused for us to decide which number that z will be sent to after we take square root of z. This confusion comes from the multiple value of argument. In fact to represent a complex number in polar coordinate, we can write $z = re^{i\theta}$. Then a square root of z can be represented by $\sqrt{r}e^{i\theta/2}$. From this expression, r is unique determined. However θ is not. It takes multiple values. But we can still restrict θ, for example in its principal range. Then the θ can now be uniquely determined. Therefore generally $z^{1/2}$ is not a function. But if we define

$$z^{1/2} := \sqrt{|z|}e^{i\mathrm{Arg}(z)/2},$$

then the value of $z^{1/2}$ can be uniquely determined. Since this $z^{1/2}$ is defined in terms of the principal argument, this function is also called principal square-root function. More generally we can define

$$z^{1/2} := \sqrt{|z|}e^{i\theta/2}, \qquad \text{with } \theta \in (\theta_0, \theta_0 + 2\pi]. \tag{1.16}$$

Here θ_0 is an arbitrary number in $\mathbb{R}$. Since there is only one argument of a given $z \neq 0$ lying on the interval $(\theta_0, \theta_0 + 2\pi]$, the $z^{1/2}$ in (1.16) is still a function. From the above descriptions, we know that to define a square root function, we must assign its argument range. The range of argument, i.e. $(\theta_0, \theta_0 + 2\pi]$ is referred as branch of $z^{1/2}$ in the following. For any $z \neq 0$, $z^{1/2}$ can be defined in terms of (1.16). At $z = 0$, by Example 12 in Sect. 3, the square-root of 0 are all zero. Therefore $z^{1/2}$ in (1.16) is well-defined on whole $\mathbb{C}$.

Example 6. Given $a_0 \in \mathbb{C}$, $T(z) := z + a_0$ is called translation function. Given a $\theta_0 \in \mathbb{R}$, $\mathrm{Rot}_{\theta_0}(z) := e^{i\theta_0}z$ is called rotation function. Given $r_0 > 0$, $S_{r_0}(z) := r_0 z$ is called scaling function. Domains of these functions are all $\mathbb{C}$.

Example 7. Exponential function. Given any complex number c, we can evaluate e^{cz} for any given $z \in \mathbb{C}$. This function e^{cz} will be called exponential function. Domain is $\mathbb{C}$.

Example 8. For real number θ, we know

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

With the definition of exponential function in Example 7, we can extend the definition of cos and sin to complex numbers. In fact we define

$$\cos z := \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z := \frac{e^{iz} - e^{-iz}}{2i}.$$

We can also define hyperbolic sine and hyperbolic cosine function as follows:

$$\sinh z := \frac{e^{z} - e^{-z}}{2}, \qquad \cosh z := \frac{e^{z} + e^{-z}}{2}.$$

Domains of functions in this example are $\mathbb{C}$.

Example 9. We can also consider the inverse function of e^z. Given a z, there is unique number w which equals to e^z. Now we consider the following question. If we are given a w, can we find a z so that $e^z = w$? Suppose $w = \rho_0 e^{i\theta_0}$. $z = x + iy$. Then it holds

$$e^{x+iy} = e^x e^{iy} = \rho_0 e^{i\theta_0}.$$

Therefore we must have

$$e^x = \rho_0 \qquad \text{and} \qquad e^{iy} = e^{i\theta_0}. \tag{1.17}$$

The first equation in (1.17) gives us

$$x = \ln\rho_0, \qquad \text{provided that } \rho_0 = |w| \neq 0.$$

To solve y, we will have problems. In fact by the second equation in (1.17), y must satisfy

$$e^{i\left(y-\theta_0\right)} = 1.$$

Hence it holds

$$y = \theta_0 + 2k\pi, \quad k \in \mathbb{Z}.$$

Here comes the multiple-value problem again. Like the square-root function case, in order to fix a unique value for y, we also need to fix the range of y. For example given an arbitrary $\alpha_0 \in \mathbb{R}$, we force y to be in the interval

$(\alpha_0, \alpha_0 + 2\pi]$. Then we can find a unique k so that $\theta_0 + 2k\pi \in (\alpha_0, \alpha_0 + 2\pi]$. Therefore $x + iy$ can be uniquely solved. Motivated by the above arguments, we define

$$\log z := \ln |z| + i\theta, \qquad \text{where } \theta \text{ is the argument of } z \text{ in the interval } (\alpha_0, \alpha_0 + 2\pi]. \tag{1.18}$$

The function $\log z$ is called logarithm function. The range of arguments, i.e. $(\alpha_0, \alpha_0 + 2\pi]$, is called branch of the log function. To define a log function, it must be accompanied with an assignment of branch. Pay attention to (1.18). The real part of $\log z$ is $\ln |z|$. It has no definition at $z = 0$. Therefore $\log z$ is defined for all points on $\mathbb{C}$ except 0. Now we calculate $\log i$ for example. Without assigning any branch, the notation $\log i$ means a set of all numbers which equal to i after taking exponential. By the calculation above, it follows

$$\log i = \ln |i| + i\arg(i) = i\left(\frac{\pi}{2} + 2k\pi\right), \qquad k \in \mathbb{Z}.$$

If we restrict principal branch for log function, then the log function is usually denoted by $\mathrm{Log} z$. Therefore $\mathrm{Log} i = i\frac{\pi}{2}$.

Example 10. With the definition of exponential function and log function, we can also define general power function. For any given complex number c, we define

$$z^c := e^{c \log z}.$$

Since log function should be defined with a given branch, z^c can also be defined with a branch a-priorily assigned. **Generally for any given c, z^c might not be able to be defined at** 0. See Example 3 in the next section. Same as the log function, if z^c is not assigned any branch, then it denotes a set of numbers which can be represented by $e^{c \log z}$. For example i^i, it holds

$$i^i = e^{i \log i} = e^{-\left(2k + \frac{1}{2}\right)\pi}.$$

If we use principal branch to define z^c, then the power function is called principal power function. In this case i^i is a unique number which equals to $e^{-\pi/2}$.

Sect. 5. Continuity of a Function. Suppose that Ω is an open set of $\mathbb{C}$. f is a complex-valued function defined on Ω. Surely f can be represented by $f(z) = f_1(z) + if_2(z)$ where f_1 and f_2 are two real-valued functions on Ω. Therefore we can use concepts of continuity for real-valued functions to define continuity of a complex-valued function.

Definition 1.16 (Continuity)**.** *Suppose $z_0 \in \Omega$, then $f = f_1 + f_2 i$ is continuous at z_0 if and only if f_1, f_2 are continuous at z_0. If f is continuous at all points in Ω, then we call f is continuous on Ω.*

With this definition, the following properties are standard for complex-valued functions.

Proposition 1.17. *Suppose f and g are continuous functions on Ω. Here Ω is an open set in $\mathbb{C}$. Then $f \pm g$, fg are also continuous functions on Ω. Moreover f/g is also continuous on Ω except possibly at the points on which $g = 0$.*

We can also compose two continuous functions together and obtain

Proposition 1.18. *Let $f : \Omega_1 \longrightarrow \Omega_2$, $g : \Omega_2 \longrightarrow \Omega_3$ be two continuous functions. Then $g \circ f$ is a continuous function from Ω_1 to Ω_3. Here Ω_1, Ω_2 and Ω_3 are three open sets in $\mathbb{C}$.*

Now we take a look at some examples.

Example 1. $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, $|z|$, $\overline{z}$ are all continuous functions. If f is a continuous function on some open set Ω, then $|f(z)|$ is also a continuous function on Ω.

Example 2. Check if the function

$$f(z) := \begin{cases} z/\overline{z}, & \text{if } z \neq 0; \\ 1, & \text{if } z = 0 \end{cases}$$

is continuous at 0.

Solution: For $z = x + iy \neq 0$, the real part of this f is read as

$$\mathrm{Re}(f) = \frac{x^2 - y^2}{x^2 + y^2}$$

If we approach the origin along $(x, 0)$, then $\mathrm{Re}(f)(x) = 1$ for all x. If we approach the origin along $(0, y)$, then $\mathrm{Re}(f)(iy) = -1$ for all y. Clearly $\mathrm{Re}(f)$ has no limit while (x, y) approaches to 0. Therefore no matter what value we assign for f at the origin, f can never be continuous at 0.

We remark here that same as for the real-valued case, f is continuous at z_0 iff

$$\lim_{z \to z_0} f(z) = f(z_0).$$

If we write $f = u + iv$, then u and v must have same limit no matter how we approach z_0. Moreover the limit should equal to the value of f at z_0.

Example 3. Check for what c the function

$$f(z) := \begin{cases} z^c, & \text{if } z \neq 0; \\ 0, & \text{if } z = 0. \end{cases}$$

is continuous at 0. Here z^c is defined on the branch $(-\pi, \pi]$.

Solution: To consider this problem, we first recall (1.18) and the definition of z^c in Example 10 of Sect. 4. In fact for all $z \neq 0$, we have

$$z^c = e^{c \log z} = e^{c\left(\ln|z| + i\theta\right)}, \qquad \text{where } \theta \in (-\pi, \pi].$$

Notice that the last equality above holds by (1.18). Suppose that $c = c_1 + c_2 i$ where c_1 and c_2 are two real values. Then the above equality can be reduced to

$$z^c = e^{\left(c_1 \ln|z| - c_2\theta\right) + i\left(c_1\theta + c_2 \ln|z|\right)} = |z|^{c_1} e^{-c_2\theta} e^{i\left(c_1\theta + c_2 \ln|z|\right)}.$$

Case 1. $c_1 = 0$**.** In this case $f(z) = z^c = e^{-c_2\theta} e^{i\left(c_2 \ln|z|\right)}$ for all $z \neq 0$. We take a modulus of $f(z)$ and obtain

$$|f(z)| = |z^c| = e^{-c_2\theta}, \qquad \text{for all } z \neq 0. \tag{1.19}$$

If f is continuous at 0, then so is $|f(z)|$. Therefore as z approaches to 0, $|f|$ should have limit. By the definition in this example the limit must be 0. However if $c_2 \neq 0$, then we can approach 0 along the ray with angle 0. Therefore on this ray, $\theta = 0$. By (1.19), it holds $|f(z)| = 1$ for all points on the ray with angle 0. We can also let z approach 0 along the ray with angle $\pi/2$. Still by (1.19), $|f(z)| = e^{-c_2\pi/2}$ for all points on the ray with angle $\pi/2$. If $c_2 \neq 0$, then when we approach 0 along two different ways as above, $|f|$ will take different limit. This is a contradiction if we assume that f is continuous at 0. Therefore if $c_2 \neq 0$, then f can not be continuous at 0. If $c_2 = 0$, by (1.19) we have $|f| = 1$ for all θ. This still implies the discontinuity of f at the origin since by definition $f = 0$ at the origin. If f is continuous at the origin that the limit of $|f|$ as z goes to 0 should be 0. But in the case here $|f| = 1$ for all $z \neq 0$ provided that $c_1 = c_2 = 0$.

Case 2. $c_1 < 0$**.** In this case it holds

$$|f(z)| = |z|^{c_1} e^{-c_2\theta}, \qquad \text{for all } z \neq 0.$$

Since $c_1 < 0$, for any fixed θ, it holds

$$\lim_{z\to 0} |f(z)| = +\infty.$$

Therefore f cannot be continuous at 0.

Case 3. $c_1 > 0$**.** In this case it holds

$$|f(z)| = |z|^{c_1} e^{-c_2\theta}, \qquad \text{for all } z \neq 0.$$

Since $c_1 > 0$, for any fixed θ, it holds

$$\lim_{z\to 0} |f(z)| = 0.$$

Clearly f is continuous at 0.

In summary, only for all c with $\mathrm{Re}(c) > 0$, the function f is a continuous function at 0.

Example 4. Let $z^{1/2}$ be the principal square-root function. Then it is discontinuous on $\Big\{(x, 0) : x < 0\Big\}$. In fact we have

$$z^{1/2} = \sqrt{|z|}\, e^{i\mathrm{Arg}(z)\big/2}$$

Consider the unit circle in $\mathbb{C}$ with center 0 and denote by z_0 the point $(-1, 0)$. If z is on the upper-circle and keeps close to z_0, then its principal arguments is positive and approach to π as z approach to z_0 from the upper-half part of the circle. In this case the limit of $z^{1/2}$ equals to $e^{i\pi/2}$. If z is on the lower-half part of the circle and keeps close to z_0, then its principal arguments is negative and approach to $-\pi$ as z approach to z_0 from the lower-half part of the circle. In this case the limit of $z^{1/2}$ equals to $e^{-i\pi/2}$. Other points on the negative half part of the x-axis can be similarly studied.

Sect.6. Differentiability of a function and Cauchy-Riemann Equation

Throughout this section, Ω denotes an open set in $\mathbb{C}$. f is a complex-valued function defined on Ω. To emphasize its real and imaginary parts, we also represent f by $f = u + iv$, where u and v are real-valued functions on Ω.

Differentiability at a given point $z_0 \in \Omega$**.** The method to define differentiability of a function at a given point $z_0 \in \Omega$ is similar to the way that we have used in the real-valued function case. Since Ω is an open set and $z_0 \in \Omega$, we can find a $r_0 > 0$ so that $D(z_0; r_0) \subset \Omega$. For any $z \in D(z_0; r_0)$ and $z \neq z_0$, we can construct a ratio

$$\frac{f(z) - f(z_0)}{z - z_0}, \qquad \text{for all } z \in D(z_0; r_0) \text{ and } z \neq z_0.$$

Then we call f is derivable/differentiable at z_0 if the limit

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{1.20}$$

exists. As for the real case, we also denote by $f'(z_0)$ the derivative of f at z_0. That is

$$f'(z_0) := \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Since we can split a complex-valued function into its real and imaginary part, limit in (1.20) exists means that both the real and imaginary parts of $\dfrac{f(z)-f(z_0)}{z-z_0}$ have limits as $(x,y)\to(x_0,y_0)$. Here we let $z=x+iy$ and $z_0=x_0+iy_0$.

Example 1. Suppose that $f(z)=1/z$. At each non-zero point z_0, we have

$$\frac{f(z)-f(z_0)}{z-z_0}=\frac{\frac{1}{z}-\frac{1}{z_0}}{z-z_0}=\frac{z_0-z}{zz_0}\frac{1}{z-z_0}=-\frac{1}{zz_0}. \tag{1.21}$$

Since $1/z$ is a continuous function at $z_0\neq 0$, it holds

$$\lim_{z\to z_0}\frac{1}{z}=\frac{1}{z_0}.$$

Applying this limit to (1.21) yields

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=-\lim_{z\to z_0}\frac{1}{zz_0}=-\frac{1}{z_0^2}.$$

Therefore f is derivable at $z_0\neq 0$. $f'(z_0)=-z_0^{-2}$.

Example 2. If $f(z)=\overline{z}$, then for any z_0, we have

$$\frac{f(z)-f(z_0)}{z-z_0}=\frac{\overline{z}-\overline{z_0}}{z-z_0}. \tag{1.22}$$

Letting $w=z-z_0$, we know that while $z\to z_0$, it should have $w\to 0$. Therefore if the right-hand side of (1.22) has limit as $z\to z_0$, then equivalently the following limit should also exists:

$$\lim_{w\to 0}\frac{\overline{w}}{w}. \tag{1.23}$$

But this is impossible. Since if w is real, then $\overline{w}=w$. It follows $\overline{w}/w=1$. If w is pure imaginary number, then $\overline{w}=-w$. It follows $\overline{w}/w=-1$. Therefore the limit in (1.23) does not exist.

Example 3. Consider the function $f(z)=|z|^2$. given any z_0, we calculate

$$\frac{f(z)-f(z_0)}{z-z_0}=\frac{|z|^2-|z_0|^2}{z-z_0} \tag{1.24}$$

If we let $w=z-z_0$, then $|z|^2=|z_0+w|^2=(z_0+w)(\overline{z_0}+\overline{w})=|z_0|^2+w\overline{w}+w\overline{z_0}+z_0\overline{w}$. Plugging this calculation into (1.24) then yields

$$\frac{f(z)-f(z_0)}{z-z_0}=\frac{|z|^2-|z_0|^2}{z-z_0}=\frac{w\overline{w}+w\overline{z_0}+z_0\overline{w}}{w}=\overline{w}+\overline{z_0}+z_0\frac{\overline{w}}{w}. \tag{1.25}$$

If $z_0=0$, then we have from (1.25) that

$$\frac{f(z)-f(0)}{z-0}=\overline{w}.$$

Therefore it holds

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}=\lim_{w\to 0}\overline{w}=0.$$

This implies the differentiability of $|z|^2$ at 0. But if $z_0\neq 0$, the last term in (1.25), that is $z_0\dfrac{\overline{w}}{w}$ has no limit as $w\to 0$. Therefore (1.25) has no limit as $w=z-z_0\to 0$, provided that $z_0\neq 0$. This tells us that $|z|^2$ is not differentiable at $z_0\neq 0$.

Remark 1.19. *Example 3 illustrates the following two facts.*
(a). A function $f = u + iv$ can be differentiable at a single point but nowhere else in any neighborhood of that points;
(b). Since $u = x^2 + y^2$ and $v = 0$ when $f(z) = |z|^2$, one can see that the real and imaginary parts of a function of a complex variable can have continuous partial derivatives of all orders at a point and yet the function of z may not be differentiable there.

Since the definition of derivative in the complex case is similar to the one given in real-valued function case, the following rules for differentiation are still held in complex case.
(1). If f and g are two complex functions, a and b are two complex numbers, then it holds

$$(af + bg)' = af' + bg'.$$

(2). If f and g are two complex functions, then it holds

$$(fg)' = f'g + fg'.$$

(3). If f and g are two complex functions, then it holds

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}.$$

(4). If f and g are two complex functions, then it holds

$$f(g(z))' = f'(g(z))g'(z).$$

Here we assume f and g are all derivable functions.

Now we try to explain why (b) in Remark 1.19 can happen. Supposing that f is derivable at z_0, then we know that the limit in (1.20) must exist. Therefore if we approach $z_0 = x_0 + iy_0$ horizontally or vertically, the limits obtained should be a unique one. More precisely we let $z = (x_0 + h) + iy_0$, where h is a real number. Then we can write

$$\begin{aligned}\frac{f(z) - f(z_0)}{z - z_0} &= \frac{u(x_0 + h, y_0) + iv(x_0 + h, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{h} \\ &= \frac{u(x_0 + h, y_0) - u(x_0, y_0)}{h} + i\frac{v(x_0 + h, y_0) - v(x_0, y_0)}{h}\end{aligned}$$

Now we let $h \to 0$ and get

$$f'(z_0) = \partial_x u\big|_{(x_0,y_0)} + i\partial_x v\big|_{(x_0,y_0)}. \tag{1.26}$$

Here we have used the definition of partial derivatives for a real-valued function at (x_0, y_0). If we let $z = x_0 + i(y_0 + h)$, where h is a real number. Then we can write

$$\begin{aligned}\frac{f(z) - f(z_0)}{z - z_0} &= \frac{u(x_0, y_0 + h) + iv(x_0, y_0 + h) - u(x_0, y_0) - iv(x_0, y_0)}{ih} \\ &= \frac{v(x_0, y_0 + h) - v(x_0, y_0)}{h} - i\frac{u(x_0, y_0 + h) - u(x_0, y_0)}{h}\end{aligned}$$

Now we let $h \to 0$ and get

$$f'(z_0) = \partial_y v\big|_{(x_0,y_0)} - i\partial_y u\big|_{(x_0,y_0)}. \tag{1.27}$$

Using (1.26) and (1.27), we get

$$\partial_x u\big|_{(x_0,y_0)} = \partial_y v\big|_{(x_0,y_0)}, \qquad \partial_x v\big|_{(x_0,y_0)} = -\partial_y u\big|_{(x_0,y_0)}. \tag{1.28}$$

In other words if f is derivable at $z = z_0$, then not only should we have the first order partial derivatives of its real and imaginary parts. But also u and v should satisfy (1.28). (1.28) is a god-given system and will be referred as Cauchy-Riemann equation in the following. We summarize the above results as follows.

Theorem 1.20. *Suppose that $f = u + iv$ and that $f'(z)$ exits at a point $z_0 = x_0 + iy_0$. then the first-order partial derivatives of u and v must exist at (x_0, y_0), and they must satisfy the Cauchy-Riemann equation (1.28). Also $f'(z_0)$ can be written as*

$$f'(z_0) = \partial_x u\big|_{(x_0,y_0)} + i\partial_x v\big|_{(x_0,y_0)}.$$

Notice that the above theorem only shows that if f is derivable at $z = z_0$, then its real and imaginary parts should satisfy Cauchy-Riemann equation (1.28). Conversely we cannot simply conclude the derivability of f at a point $z = z_0$ by the satisfaction of Cauchy-Riemann equation.

Example 4. Consider

$$f(z) = \begin{cases} \overline{z}^2/z, & \text{when } z \neq 0; \\ 0, & \text{when } z = 0. \end{cases}$$

Show that f satisfies Cauchy-Riemann equation at $z = 0$. But f is not derivable at $z = 0$.

Solution: When $(x, y) \neq (0, 0)$, the real and imaginary parts of f are read as

$$u(x, y) = \frac{x^3 - 3xy^2}{x^2 + y^2} \qquad \text{and} \qquad v(x, y) = \frac{y^3 - 3x^2y}{x^2 + y^2},$$

respectively. Also, $u(0, 0) = 0$ and $v(0, 0) = 0$. Because

$$u_x(0, 0) = \lim_{h\to 0} \frac{u(h, 0) - u(0, 0)}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

and

$$v_y(0, 0) = \lim_{h\to 0} \frac{v(0, y) - v(0, 0)}{h} = \lim_{h\to 0} \frac{h}{h} = 1,$$

we find that the first Cauchy-Riemann equation $u_x = v_y$ is satisfied at $z = 0$. Likewise, it is easy to show that $u_y = 0 = -v_x$ at $z = 0$. But

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z\to 0} \left(\frac{\overline{z}}{z}\right)^2$$

does not exist. The reason is that we can assume $z = \rho e^{i\theta_0}$ where θ_0 is fixed and $\rho \to 0$. Plugging into right-hand side above yield

$$\left(\frac{\overline{z}}{z}\right)^2 = e^{-4i\theta_0}.$$

Of course this quantity depends on the angle of ray where z point lies on.

From Example 4, we know that Cauchy-Riemann equation is only a necessary condition to allow f derivable at z_0. We can not simply imply the differentiability of f at z_0 just because f satisfies Cauchy-Riemann equation at z_0. In order to imply that f is derivable at z_0, extra condition (besides Cauchy-Riemann equation) should be added.

Theorem 1.21. *Let the function $f(z) = u(x, y) + iv(x, y)$ be defined throughout some $D(z_0; \epsilon)$, and suppose that*

(a). the first-order partial derivatives of the functions u and v with respect to x and y exist everywhere in the $D(z_0;\epsilon)$;
(b). those partial derivatives are continuous at (x_0,y_0) and satisfy the Cauchy-Riemann equations at (x_0,y_0), Then $f'(z_0)$ exists, its value being $f'(z_0)=u_x(x_0,y_0)+iv_x(x_0,y_0)$.

Proof. Fixing an $z_0\in\Omega$, for z sufficiently close to z_0, we can use a segment parameterized by $z(t)=tz+(1-t)z_0$ ($t\in[0,1]$) to connect z_0 and z. Therefore for $f(z)$ satisfying the assumptions in the theorem, it follows

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}f(z(t)) &= \partial_x f\Big|_{z(t)}(x-x_0)+\partial_y f\Big|_{z(t)}(y-y_0)\\ &= \partial_x f\Big|_{z_0}(x-x_0)+\partial_y f\Big|_{z_0}(y-y_0)+\left(\partial_x f\Big|_{z(t)}-\partial_x f\Big|_{z_0}\right)(x-x_0)+\left(\partial_y f\Big|_{z(t)}-\partial_y f\Big|_{z_0}\right)(y-y_0).\end{aligned}$$

Assume that $f=u+iv$. Then

$$\begin{aligned}\partial_x f\Big|_{z_0}(x-x_0)+\partial_y f\Big|_{z_0}(y-y_0) &= \partial_x u\Big|_{z_0}(x-x_0)+i\partial_x v\Big|_{z_0}(x-x_0)+\partial_y u\Big|_{z_0}(y-y_0)+i\partial_y v\Big|_{z_0}(y-y_0)\\ &= \underbrace{\partial_x u\Big|_{z_0}(x-x_0)+i\partial_x v\Big|_{z_0}(x-x_0)-\partial_x v\Big|_{z_0}(y-y_0)+i\partial_x u\Big|_{z_0}(y-y_0)}_{\text{Cauchy}-\text{Riemann Equation}}\\ &= \left(\partial_x u\Big|_{z_0}+i\partial_x v\Big|_{z_0}\right)(z-z_0)\,.\end{aligned}$$

The above two equations and fundamental theorem of calculus infer

$$\begin{aligned}f(z)-f(z_0)=\int_0^1\frac{\mathrm{d}}{\mathrm{d}t}f(z(t))\,\mathrm{d}t &= \left(\partial_x u\Big|_{z_0}+i\partial_x v\Big|_{z_0}\right)(z-z_0)\\ &+ \int_0^1\left(\partial_x f\Big|_{z(t)}-\partial_x f\Big|_{z_0}\right)(x-x_0)+\left(\partial_y f\Big|_{z(t)}-\partial_y f\Big|_{z_0}\right)(y-y_0)\,\mathrm{d}t.\end{aligned}$$

Now we divide $z-z_0$ from both sides above and yield

$$\frac{f(z)-f(z_0)}{z-z_0}-\left(\partial_x u\Big|_{z_0}+i\partial_x v\Big|_{z_0}\right)=\int_0^1\left(\partial_x f\Big|_{z(t)}-\partial_x f\Big|_{z_0}\right)\left(\frac{x-x_0}{z-z_0}\right)+\left(\partial_y f\Big|_{z(t)}-\partial_y f\Big|_{z_0}\right)\left(\frac{y-y_0}{z-z_0}\right)\,\mathrm{d}t$$

Therefore it holds

$$\begin{aligned}\left|\frac{f(z)-f(z_0)}{z-z_0}-\left(\partial_x u\Big|_{z_0}+i\partial_x v\Big|_{z_0}\right)\right| &\leqslant \int_0^1\left|\partial_x f\Big|_{z(t)}-\partial_x f\Big|_{z_0}\right|\underbrace{\left|\frac{x-x_0}{z-z_0}\right|}_{\leqslant 1}+\left|\partial_y f\Big|_{z(t)}-\partial_y f\Big|_{z_0}\right|\underbrace{\left|\frac{y-y_0}{z-z_0}\right|}_{\leqslant 1}\,\mathrm{d}t\\ &\leqslant \max_{t\in[0,1]}\left|\partial_x f\Big|_{z(t)}-\partial_x f\Big|_{z_0}\right|+\max_{t\in[0,1]}\left|\partial_y f\Big|_{z(t)}-\partial_y f\Big|_{z_0}\right|\end{aligned}$$

If $z\to z_0$, then all $z(t)$ with $t\in[0,1]$ converges to z_0 also. Therefore condition b induces

$$\max_{t\in[0,1]}\left|\partial_x f\Big|_{z(t)}-\partial_x f\Big|_{z_0}\right|+\max_{t\in[0,1]}\left|\partial_y f\Big|_{z(t)}-\partial_y f\Big|_{z_0}\right|\longrightarrow 0 \qquad \text{as } z\to z_0.$$

The proof is done. □

Example 5. Consider the function $f(z)=e^z=e^x\cos y+ie^x\sin y$. Its $u(x,y)=e^x\cos y$ and $v(x,y)=e^x\sin y$. Then simply calculations yield

$$u_x=e^x\cos y=v_y \qquad \text{and} \qquad u_y=-e^x\sin y=-v_x.$$

Obviously all assumptions in Theorem 1.21 are satisfied and we have $f'(z) = u_x + iv_x = e^x \cos y + ie^x \sin y = e^z$.

Example 6. When using the Theorem 1.21 to find a derivative at z_0, one must be careful not to use the expression for $f'(z)$ in the statement of the theorem before the existence of $f'(z)$ at z_0 is established. Consider, for instance, the function

$$f(z) = x^3 + i(1-y)^3.$$

Here $u(x,y) = x^3$ and $v(x,y) = (1-y)^3$. It would be a mistake to say that $f'(z)$ exists everywhere and that $f'(z) = u_x + iv_x = 3x^2$. To see this, we observe that the first C-R equation $u_x = v_y$ can hold only if $x^2 + (1-y)^2 = 0$. The second C-R equation $u_y = -v_x$ is always satisfied. Therefore C-R equation is only satisfied at $x = 0$ and $y = 1$. Therefore we know that only at $z = i$, $f'(z)$ exists, in which case $f'(i) = 0$.

Sect.7 Analyticity and Harmonicity We are now ready to introduce the concept of an analytic function.

Definition 1.22. *A function f of the complex variable z is* ***analytic in an open set*** *S if it has a derivative everywhere in S. It is* ***analytic at a point*** *z_0 if it is analytic on $D(z_0;\epsilon)$ for some $\epsilon > 0$.*

Remark 1.23. *If a set S is not an open set, we can still define analyticity for a function f defined on S. In this case we call f is analytic on S if there is an open set O containing S so that f is analytic on O.*

Example 1. Recall Example 1 in Sect. 6, it is clear that $1/z$ is analytic at all non-zero points. The reason is $\mathbb{C}\backslash\{0\}$ is an open set and by Example 1 in Sect. 6, $f'(z) = -z^{-2}$ exists for all $z \neq 0$. Recall Example 3 in Sect. 6, $|z|^2$ is only derivable at $z = 0$. for any $D(0;r)$, $|z|^2$ is not analytic on $D(0;r)$. Therefore though $|z|^2$ is derivable at 0 but it is not analytic at 0.

In the following we consider an important property of analytic function. Before proceeding we need a definition on path-connected open set.

Definition 1.24. *An open set Ω is called path-connected if for any two points P and P' in Ω, there is a differentiable curve l, parameterized by $(x(t), y(t))$ with $t \in [a,b]$, so that $x(t)$ and $y(t)$ are differentiable functions on (a,b). Meanwhile $x(t)$ and $y(t)$ are continuous at $t = a$ and $t = b$ with $P = (x(a), y(a))$ and $P' = (x(b), y(b))$.*

With Definition 1.24, we have

Theorem 1.25. *Suppose that Ω is a path-connected open set. If $f'(z) = 0$ everywhere in Ω, then $f(z)$ must be constant throughout Ω.*

Proof. Let $f = u(x,y) + iv(x,y)$. By Cauchy-Riemann equation, it holds $f'(z) = u_x + iv_x = v_y - iu_y = 0$. Therefore we have

$$u_x = u_y = v_x = v_y = 0, \qquad \text{on } \Omega. \tag{1.29}$$

Fixing a point P_0 in Ω, for any $P \in \Omega$, we denote by $(x(t), y(t))$ a parametrization of a curve connecting P_0 and P. This curve, denoted by l, is contained in Ω. Now we consider the restriction of u on l. that is the function $u(x(t), y(t))$. Simple calculation yields

$$\frac{\mathrm{d}}{\mathrm{d}t} u(x(t), y(t)) = \partial_x u\Big|_{(x(t),y(t))} x'(t) + \partial_y u\Big|_{(x(t),y(t))} y'(t).$$

by (1.29), the above equality implies

$$\frac{\mathrm{d}}{\mathrm{d}t} u(x(t), y(t)) = 0, \qquad \text{for any } t \in [a,b].$$

therefore it holds $u(x(t), y(t)) = u(x(a), y(a)) = u(x(b), y(b))$ for any $t \in [a,b]$. since l connects P_0 and P, it follows $u(P_0) = u(P)$. Notice that P is an arbitrary point in Ω. u is a constant function. Same arguments can be applied to v. Finally f is a constant function in Ω. ☐

Example 2. Let Ω be a path-connected open set in $\mathbb{C}$. Suppose that f is an analytic function on Ω. If f and $\overline{f}$ are all analytic on Ω, then f must be a constant.

Solution: Assume that $f = u + iv$. Since f is analytic in Ω, u and v satisfy

$$u_x = v_y, \qquad u_y = -v_x. \tag{1.30}$$

Since $\overline{f} = u - iv$ is also analytic on Ω, it holds

$$u_x = -v_y, \qquad u_y = v_x. \tag{1.31}$$

(1.30)-(1.31) imply $u_x = u_y = v_x = v_y = 0$ on Ω. Therefore $f'(z) = u_x + iv_x = 0$ on Ω. By Theorem 1.25, f must be a constant.

Example 3. Let Ω be a path-connected open set in $\mathbb{C}$. Suppose that f is an analytic function on Ω. If $|f|$ is a constant function on Ω, then f must also be a constant function on Ω.

Solution: If $|f| = 0$ on Ω, then obviously $f = 0$ on Ω. Now we assume $|f| = c \neq 0$ on Ω. Since $|f|^2 = f\overline{f} = c^2$, it follows

$$\overline{f} = \frac{c^2}{f}.$$

By analyticity of f, we know that $\overline{f}$ is also analytic on Ω. The last example implies that f is a constant function.

Example 4. Let Ω be an open set in $\mathbb{C}$. Suppose that $f = u + iv$ is an analytic function on Ω. Then u and v are all harmonic functions on Ω.

Solution: Since u and v satisfies Cauchy-Riemann equation, it holds $u_x = v_y$. Furthermore we get $u_{xx} = v_{xy}$. We also have $u_y = -v_x$. Therefore $u_{yy} = -v_{xy}$. From these computations, we get $u_{xx} + u_{yy} = v_{xy} - v_{xy} = 0$. That is $\Delta u = 0$ on Ω. In other words u is a harmonic function on Ω. The arguments for v is similar.

Sect. 8 Integrals Starting from this section, we study integral theory for complex valued function. For simplicity, we call Ω a domain in $\mathbb{C}$ if Ω is a path-connected open set in $\mathbb{C}$. Without mentioning Ω in the following are all domains in $\mathbb{C}$. Moreover a curve l is called Jordan curve if it is a continuous curve without self-intersection. Notice that a Jordan curve l is in fact only a geometric object. It has no direction. But if we parameterize it by a parametrization $(x(t), y(t))$ with $t \in [a, b]$, then automatically it is assigned a direction. In fact when the parameter t increase from a to b, the parametrization $(x(t), y(t))$ sweeps out the points on l from the initial point $(x(a), y(a))$ to the end-point $(x(b), y(b))$. In this case, the parametrization induces a direction on the Jordan curve l. In the following integrals on a Jordan curve will be defined in terms of a parametrization of this curve. Therefore all Jordan curves in the following arguments should be understood as a directional curve with direction induced by a parametrization.

Integral along path Let l be a Jordan curve in Ω. Suppose that $(x(t), y(t))$ with $t \in [a, b]$ is a parametrization of l. If f is a continuous function on Ω, then we can restrict f on l to get

$$f\big(x(t) + iy(t)\big), \qquad t \in [a, b].$$

Let $f = u + iv$. Then we have

$$f\big(x(t) + iy(t)\big) = u\big(x(t), y(t)\big) + iv\big(x(t), y(t)\big). \tag{1.32}$$

Noticing that the real and imaginary parts on the right-hand side of (1.32) are all real-valued single variable functions. Therefore we can use definition of single variable real-valued functions to define the integral of f. More precisely we define

$$\int_a^b f\big(x(t) + iy(t)\big)\,\mathrm{d}t := \int_a^b u\big(x(t), y(t)\big)\,\mathrm{d}t + i\int_a^b v\big(x(t), y(t)\big)\,\mathrm{d}t.$$

All techniques used in the calculations of integrals for real-valued single variable functions can be applied in the complex scenario.

Example 1. Compute

$$\int_0^{\pi/4} e^{it}\mathrm{d}t = \int_0^{\pi/4} \cos t + i\sin t\mathrm{d}t = \int_0^{\pi/4} \cos t\mathrm{d}t + i\int_0^{\pi/4} \sin t\mathrm{d}t = \sin t\Big|_0^{\pi/4} + i(-\cos t)\Big|_0^{\pi/4} = \frac{\sqrt{2}}{2} + i\left(1 - \frac{\sqrt{2}}{2}\right).$$

The fundamental theorem of calculus in the real-case still holds in the complex case. More precisely let

$$F\big(x(t) + iy(t)\big) = U\big(x(t), y(t)\big) + iV\big(x(t), y(t)\big).$$

Then we can compute[1]

$$\frac{\mathrm{d}}{\mathrm{d}t}F\big(x(t) + iy(t)\big) = \frac{\mathrm{d}}{\mathrm{d}t}U\big(x(t), y(t)\big) + i\frac{\mathrm{d}}{\mathrm{d}t}V\big(x(t), y(t)\big).$$

If $f = u + iv$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}U\big(x(t), y(t)\big) = u\big(x(t), y(t)\big), \qquad \frac{\mathrm{d}}{\mathrm{d}t}V\big(x(t), y(t)\big) = v\big(x(t), y(t)\big),$$

then we have

$$\begin{aligned}\int_a^b f\big(x(t) + iy(t)\big)\,\mathrm{d}t &= \int_a^b u\big(x(t), y(t)\big)\,\mathrm{d}t + i\int_a^b v\big(x(t), y(t)\big)\,\mathrm{d}t \\ &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}U\big(x(t), y(t)\big)\,\mathrm{d}t + i\int_a^b \frac{\mathrm{d}}{\mathrm{d}t}V\big(x(t), y(t)\big)\,\mathrm{d}t \\ &= U\big(x(t), y(t)\big)\Big|_a^b + iV\big(x(t), y(t)\big)\Big|_a^b = F\big(x(t) + iy(t)\big)\Big|_a^b.\end{aligned}$$

Simply speaking, the above arguments imply

$$\int_a^b f\big(x(t) + iy(t)\big)\,\mathrm{d}t = F\big(x(t) + iy(t)\big)\Big|_a^b, \quad \text{if } \frac{\mathrm{d}}{\mathrm{d}t}F\big(x(t) + iy(t)\big) = \frac{\mathrm{d}}{\mathrm{d}t}U\big(x(t), y(t)\big) + i\frac{\mathrm{d}}{\mathrm{d}t}V\big(x(t), y(t)\big).$$

Example 2. In light of Example 1 in this section, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{e^{it}}{i} = e^{it}.$$

Therefore we easily have

$$\int_0^{\pi/4} e^{it}\,\mathrm{d}t = -ie^{it}\Big|_0^{\pi/4} = -i\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} - 1\right) = \frac{\sqrt{2}}{2} + i\left(1 - \frac{\sqrt{2}}{2}\right).$$

From the previous arguments, if we can find a $F\big(x(t) + iy(t)\big)$ so that

$$\frac{\mathrm{d}}{\mathrm{d}t}F\big(x(t) + iy(t)\big) = f\big(x(t) + iy(t)\big),$$

then the integral of $f\big(x(t) + iy(t)\big)$ on $[a, b]$ can be easily computed by making difference of $F\big(x(t) + iy(t)\big)$ at the two end-points a and b. In the real case, we have chain rule. That is

$$\big(g(x(t))\big)' = g'\big(x(t)\big)x'(t).$$

[1]for a complex valued function$f = u + iv$, if it depends on a single real parameter t, then we use $\frac{\mathrm{d}}{\mathrm{d}t}f = \frac{\mathrm{d}}{\mathrm{d}t}u + i\frac{\mathrm{d}}{\mathrm{d}t}v$. If f depends on a complex variable z, then $\frac{\mathrm{d}}{\mathrm{d}z}$ should be understood as complex derivative of f.

Therefore if function f is of the form $g'(x(t))x'(t)$, then we can easily find out its anti-derivative function. That is $g(x(t))$. In the complex case, we have similar results. Suppose $f = u + iv$ is an analytic function on Ω. Then we have[2]

$$f'(z) = u_x + i v_x. \tag{1.33}$$

Let l be parameterized by $z(t) = x(t) + iy(t)$ with $t \in [a, b]$. Then by (1.33) we have

$$f'\big(z(t)\big) = u_x\big(x(t), y(t)\big) + i v_x\big(x(t), y(t)\big). \tag{1.34}$$

Since $z'(t) = x'(t) + i y'(t)$, by (1.34) and Cauchy-Riemann equation we can calculate

$$\begin{aligned} f'\big(z(t)\big)z'(t) &= \Big(u_x\big(x(t), y(t)\big) + i v_x\big(x(t), y(t)\big)\Big)\Big(x'(t) + i y'(t)\Big) \\ &= \Big(u_x\big(x(t), y(t)\big)x'(t) - v_x\big(x(t), y(t)\big)y'(t)\Big) + i\Big(u_x\big(x(t), y(t)\big)y'(t) + v_x\big(x(t), y(t)\big)x'(t)\Big) \\ &= \Big(u_x\big(x(t), y(t)\big)x'(t) + u_y\big(x(t), y(t)\big)y'(t)\Big) + i\Big(v_y\big(x(t), y(t)\big)y'(t) + v_x\big(x(t), y(t)\big)x'(t)\Big) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\, u\big(x(t), y(t)\big) + i\,\frac{\mathrm{d}}{\mathrm{d}t}\, v\big(x(t), y(t)\big). \end{aligned}$$

In the last equality above, we have used chain rule for multiple variable functions. Summarizing the above computations yields

$$\frac{\mathrm{d}}{\mathrm{d}t} f\big(z(t)\big) = f'(z(t))z'(t). \tag{1.35}$$

Notice that (1.35) is quite similar to the chain rule for real-valued single variable functions. The only difference is $'$ for function f should be understood as derivative with respect to the complex variable z. $'$ on $z(t)$ is the standard single-variable derivative in real calculus.

Example 3. Let $f(z) = e^z$. $z(t) = it$. Here f is analytic with $f'(z) = e^z$. $z'(t) = i$. Therefore by (1.35) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{it} = i e^{it}.$$

This yields the first equality in Example 2 of this section.

Contour Integral Firstly we assume l is a differentiable Jordan curve. $z(t) = x(t) + i y(t)$ with $t \in [a, b]$ is a parametrization of l with $x(t)$ and $y(t)$ being two differentiable functions.

Definition 1.26 (Contour Integral on Differentiable Jordan Curves)**.** *Suppose l is a Jordan curve. l is parameterized by $z(t) = x(t) + iy(t)$ with $t \in [a, b]$. $f(z)$ is a complex-valued function defined on an open set Ω. l is contained in Ω. Then we let*

$$\int_l f(z)\, \mathrm{d}z := \int_a^b f\big(z(t)\big)\, z'(t)\, \mathrm{d}t. \tag{1.36}$$

From the right-hand side of (1.36), it seems that the contour integral should depend on a particular parametrization of the given curve l. But from the left-hand side of (1.36), we can not read any information about the parametrization of the curve l. Generally for a given curve l with two end-points P_1 and P_2, there can have more than one parametrization which sweeps out the curve l from P_1 to P_2. Will two parametrizations of l give two different integral results ? Suppose $z(t)$ and $w(s)$ be two parametrizations of l. Both of these two

[2]Here for single variable function, $'$ denotes the standard derivative in real calculus. If f depends on the complex variable z, then $'$ denotes its complex derivative.

parametrizations induce the same direction on l. More precisely let z be defined on $[a,b]$ and let w be defined on $[c,d]$. Moreover it satisfies

$$z(a) = w(c), \qquad \text{and} \qquad z(b) = w(d). \tag{1.37}$$

Since z and w are parametrizations of l, they are 1-1 correspondences between their associated domain intervals and the curve l. Fixing a $t \in [a,b]$, we can find a point $z(t)$ on l. By the parametrization w, we can also find a $s(t) \in [c,d]$ so that

$$w(s(t)) = z(t). \tag{1.38}$$

Therefore we obtain a change of variable function $s(t)$ which defines on $[a,b]$ and takes values in $[c,d]$. By (1.38) and the first equality in (1.37), it holds $w(s(a)) = z(a) = w(c)$. Therefore we get $s(a) = c$. Similarly by (1.38) and the second equality in (1.37), it holds $w(s(b)) = z(b) = w(d)$. This infers $s(b) = d$. The above arguments show that when t runs from a to b, $s(t)$ runs from c to d. Then we have

$$\int_c^d f\big(w(s)\big)\, w'(s)\,\mathrm{d}s \underbrace{=}_{\text{let } s = s(t)} \int_a^b f\Big(\underbrace{w(s(t))}_{\text{equals to } z(t) \text{ by (1.38)}}\Big) \underbrace{w'(s(t))\, s'(t)}_{\text{equals to } (w(s(t)))' \text{ by chain rule}} \mathrm{d}t.$$

Applying (1.38) to the right-hand side above yields

$$\int_c^d f\big(w(s)\big)\, w'(s)\,\mathrm{d}s = \int_a^b f\big(z(t)\big)\, z'(t)\,\mathrm{d}t.$$

This equality tells us that z and w determine the same integral result.

Remark 1.27. *From the above arguments, the notation l on the left-hand side of (1.36) should be understood as a directional curve. Once l is given and a direction on l is fixed, then the integration on the right-hand side of (1.36) is independent of the choice of parametrizations of the directional curve l. One can also check that if we fix l and change the direction on l (usually this directional curve is denoted by $-l$), the resulted integration on $-l$ satisfies*

$$\int_{-l} f(z)\,\mathrm{d}z = -\int_l f(z)\,\mathrm{d}z.$$

This is the same as the real calculus case. In real calculus, we have

$$\int_a^b f = -\int_b^a f.$$

So far our curves are differentiable Jordan curves. We can also extend the definition of integral to be on more general curves.

Definition 1.28 (Contour)**.** *We call l a contour if there are finitely many differentiable Jordan curves, denoted by $\Big\{l_j : j = 1, ..., N\Big\}$, so that*

$$l = \bigcup_{j=1}^N l_j. \tag{1.39}$$

Moreover for any two adjacent curves l_j and l_{j+1}, the end-point on l_j equals to the initial point on l_{j+1}.

With the definition of Contour above, we can define the general contour integration as follows:

Definition 1.29. *Suppose that f is complex-valued continuous function on the open set Ω. l is a contour contained in Ω and can be represented as in (1.39). Then we define*

$$\int_l f(z)\,\mathrm{d}z := \sum_{j=1}^{N} \int_{l_j} f(z)\,\mathrm{d}z.$$

Remark 1.30. *For a contour, we may have more than one decomposition of it. That is we can have $\{l_j : j = 1, ..., N\}$ and $\{w_j : j = 1, ..., M\}$ so that*

$$l = \bigcup_{j=1}^{N} l_j = \bigcup_{j=1}^{M} w_j.$$

However it can be easily checked that

$$\sum_{j=1}^{N} \int_{l_j} f(z)\,\mathrm{d}z = \sum_{j=1}^{M} \int_{w_j} f(z)\,\mathrm{d}z.$$

In other words the contour integration defined in Definition 1.29 is independent of the choice of the decomposition of l.

Example 4. Now we show some examples of contours.

(a). The polygonal line defined by

$$z(t) := \begin{cases} t + it, & \text{when } t \in [0, 1]; \\ t + i, & \text{when } t \in [1, 2]. \end{cases}$$

(b). The unit circle $z(\theta) = e^{i\theta}$ where $\theta \in [0, 2\pi]$ is a contour. Moreover $z(\theta) = e^{-i\theta}$ where $\theta \in [0, 2\pi]$ is also a contour. Even though the set of points on these two contours are same. However their directions are different and give us different contours. For any m an integer, we can define $z(\theta) = e^{im\theta}$ where $\theta \in [0, 2\pi]$. This is also a contour. If $m > 0$, then this contour wind around the origin m times and counter-clockwisely. If $m < 0$, then this contour wind around the origin m times and clockwisely.

Now we study some examples of contour integration.

Example 5. Let us evaluate the contour integral

$$\int_{C_1} \frac{1}{z}\,\mathrm{d}z$$

where C_1 is the top half

$$z = e^{i\theta}, \quad \theta \in [0, \pi]$$

of the circle $|z| = 1$ from $z = 1$ to $z = -1$. According to the Definition 1.26, it holds

$$\int_{C_1} \frac{1}{z}\,\mathrm{d}z = \int_0^{\pi} \frac{1}{e^{i\theta}}\, i\, e^{i\theta}\,\mathrm{d}\theta = i \int_0^{\pi} \mathrm{d}\theta = \pi i.$$

Example 6. Let C be any smooth arc with parametrization $z(t)$. Here $t \in [a, b]$. It then holds

$$\int_C z\,\mathrm{d}z = \int_a^b z(t)z'(t)\,\mathrm{d}t = \left.\frac{z^2(t)}{2}\right|_a^b = \frac{1}{2}\left(z^2(b) - z^2(a)\right).$$

Here we have used (1.35) with $f(z) = z^2/2$ there.

Example 7. Let C_1 be the contour which start from 0 to i along vertical line and then from i to $1+i$ along the horizontal line. It then holds

$$\int_{C_1} y - x - i3x^2 \, \mathrm{d}z = \int_0^1 t\, i \, \mathrm{d}t + \int_0^1 1 - t - i3t^2 \mathrm{d}t = \frac{i}{2} + \left(\frac{1}{2} - i\right).$$

Let C_2 be the contour which start from 0 and point to $1+i$ along the line $y = x$. Then it holds

$$\int_{C_2} f(z) \, \mathrm{d}z = \int_0^1 \left(t - t - i3t^2\right)(1+i) \, \mathrm{d}t = 1 - i.$$

Example 8. Let C denote the semicircular path

$$z(\theta) = 3e^{i\theta}, \qquad \theta \in [0, \pi].$$

Let $f(z) = z^{1/2}$ which is defined on the branch $0 \leqslant \arg z < 2\pi$. It then holds

$$\int_C f(z) \, \mathrm{d}z = \int_0^\pi \left(3e^{i\theta}\right)^{1/2} 3i e^{i\theta} \mathrm{d}\theta = \int_0^\pi e^{\frac{1}{2}\log\left(3e^{i\theta}\right)} 3i e^{i\theta} \mathrm{d}\theta = \int_0^\pi e^{\frac{1}{2}\left(\ln 3 + i\arg\left(3e^{i\theta}\right)\right)} 3i e^{i\theta} \mathrm{d}\theta$$

Notice that $\arg\left(3e^{i\theta}\right)$ should lie in $[0, 2\pi)$. Moreover the range of variable θ is in $[0, \pi]$ which is a subset of $[0, 2\pi)$. Therefore it holds $\arg\left(3e^{i\theta}\right) = \theta$. Plugging this computation into the last equality yields

$$\int_C f(z) \, \mathrm{d}z = 3\sqrt{3}\, i \int_0^\pi e^{3i\theta/2} \, \mathrm{d}\theta = -2\sqrt{3}(1+i).$$

Example 9. Let C denote the circle $z(\theta) = e^{i\theta}$ where $\theta \in [-\pi, \pi]$. Let $f(z) = z^{-1+i}$ where f is defined on the principal branch. It then holds

$$\int_C f(z) \, \mathrm{d}z = \int_{-\pi}^\pi \left(e^{i\theta}\right)^{-1+i} i e^{i\theta} \, \mathrm{d}\theta = \int_{-\pi}^\pi e^{(-1+i)\log e^{i\theta}} i e^{i\theta} \, \mathrm{d}\theta$$

Since θ runs within $[-\pi, \pi)$, it is in the principal branch, we have $\arg\left(e^{i\theta}\right) = \theta$. The last equality is reduced to

$$\int_C f(z) \, \mathrm{d}z = \int_{-\pi}^\pi e^{(-1+i)i\theta} i e^{i\theta} \, \mathrm{d}\theta = i \int_{-\pi}^\pi e^{-\theta} \mathrm{d}\theta = i\left(e^\pi - e^{-\pi}\right).$$

Absolute Integral Now we introduce a third type integral on curve or contour l. Suppose that l is a differentiable Jordan curve parameterized by $z(t)$ where $t \in [a, b]$. Then we define

$$\int_l f(z) \left|\mathrm{d}z\right| := \int_a^b f(z(t)) \left|z'(t)\right| \mathrm{d}t. \tag{1.40}$$

Suppose l is a contour and can be decomposed into

$$l = \bigcup_{j=1}^N l_j.$$

Here $\{l_j\}$ is a set of differentiable Jordan curves. Then we define

$$\int_l f(z) \left|\mathrm{d}z\right| := \sum_{j=1}^N \int_{l_j} f(z(t)) \left|\mathrm{d}z\right|.$$

Remark 1.31. *One can use the same arguments as for the contour integration to check that this absolute integration is independent of the direction of the curve or contour l. That is it is invariant if you change direction of the contour l.*

The following property will be frequently used in the future. It reveals a quantitative relationship between the contour integral and the absolute integral.

Theorem 1.32. *Suppose that f is a continuous function defined on an open set Ω. l is a contour contained in Ω. Then it holds*

$$\left|\int_l f(z)\,\mathrm{d}z\right| \leqslant \int_l |f(z)|\,|\mathrm{d}z|. \tag{1.41}$$

Proof. Suppose that $l = \bigcup_{j=1}^N l_j$. By triangle inequality it holds

$$\left|\int_l f(z)\,\mathrm{d}z\right| = \left|\sum_{j=1}^N \int_{l_j} f(z)\,\mathrm{d}z\right| \leqslant \sum_{j=1}^N \left|\int_{l_j} f(z)\,\mathrm{d}z\right|.$$

Therefore we can assume without loss of generality that l is a differentiable Jordan curve. If

$$\int_l f(z)\,\mathrm{d}z = 0,$$

then (1.41) automatically holds. Therefore we can assume

$$\int_l f(z)\,\mathrm{d}z \neq 0$$

and represent it by polar representation as follows:

$$\int_l f(z)\,\mathrm{d}z = \left|\int_l f(z)\,\mathrm{d}z\right| e^{i\Theta}.$$

Equivalently it follows

$$\left|\int_l f(z)\,\mathrm{d}z\right| = e^{-i\Theta}\int_l f(z)\,\mathrm{d}z.$$

Assume that $f = u + iv$ where u and v are real-valued functions on Ω. We rewrite the above equality by

$$\left|\int_l f(z)\,\mathrm{d}z\right| = e^{-i\Theta}\int_l f(z)\,\mathrm{d}z = \int_l u\cos\Theta + v\sin\Theta\,\mathrm{d}z + i\int_l v\cos\Theta - u\sin\Theta\,\mathrm{d}z.$$

Now we let $z(t) = x(t) + iy(t)$ denote a parametrization of the directional curve l. Here $t \in [a, b]$. Then the above equality is reduced to

$$\begin{aligned}
\left|\int_l f(z)\,\mathrm{d}z\right| &= \int_a^b \Big(u(x(t),y(t))\cos\Theta + v(x(t),y(t))\sin\Theta\Big)\Big(x'(t) + iy'(t)\Big)\,\mathrm{d}t \\
&+ i\int_a^b \Big(v(x(t),y(t))\cos\Theta - u(x(t),y(t))\sin\Theta\Big)\Big(x'(t) + iy'(t)\Big)\,\mathrm{d}t \\
&= \int_a^b \Big(u(x(t),y(t))\cos\Theta + v(x(t),y(t))\sin\Theta\Big)x'(t) - \Big(v(x(t),y(t))\cos\Theta - u(x(t),y(t))\sin\Theta\Big)y'(t)\,\mathrm{d}t \\
&+ i\,\text{Imaginary part}.
\end{aligned}$$

Notice that the left-hand side above is a real number. The imaginary part above must be zero. Therefore we get

$$\begin{aligned}
\left|\int_l f(z)\,\mathrm{d}z\right| &= \int_a^b u(x(t),y(t))\Big(x'(t)\cos\Theta + y'(t)\sin\Theta\Big) + v(x(t),y(t))\Big(x'(t)\sin\Theta - y'(t)\cos\Theta\Big)\,\mathrm{d}t \\
&\leqslant \int_a^b \Big|u(x(t),y(t))\Big(x'(t)\cos\Theta + y'(t)\sin\Theta\Big) + v(x(t),y(t))\Big(x'(t)\sin\Theta - y'(t)\cos\Theta\Big)\Big|\,\mathrm{d}t \\
&\leqslant \int_a^b \sqrt{u^2(x(t),y(t)) + v^2(x(t),y(t))}\sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t.
\end{aligned}$$

In the last inequality of the above, we have used Cauchy-Schwartz inequality. Since $|f|^2 = u^2 + v^2$, $|z'(t)|^2 = (x'(t))^2 + (y'(t))^2$, the last inequality is then rewritten by

$$\left| \int_l f(z)\, \mathrm{d}z \right| \leqslant \int_a^b |f(z(t))| \, |z'(t)| \, \mathrm{d}t$$

The proof then follows by (1.40). □

Now we take a look at some applications.

Example 10. Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant. Inequality (1.41) can be used to show that

$$\left| \int_C \frac{z-2}{z^4+1}\, \mathrm{d}z \right| \leqslant \int_C \left| \frac{z-2}{z^4+1} \right| |\mathrm{d}z| \leqslant \int_C \frac{|z|+2}{|z|^4-1}\, |\mathrm{d}z| = \frac{4}{15} \int_C |\mathrm{d}z|\,.$$

Since

$$\int_C |\mathrm{d}z| = \text{length of arc } C,$$

it then follows

$$\left| \int_C \frac{z-2}{z^4+1}\, \mathrm{d}z \right| \leqslant \frac{4}{15}\pi.$$

Example 11. Let C_R denote the semicircle $z(\theta) = Re^{i\theta}$ where $\theta \in [0, \pi]$. It holds

$$\left| \int_{C_R} \frac{z+1}{(z^2+4)(z^2+9)}\, \mathrm{d}z \right| \leqslant \int_{C_R} \left| \frac{z+1}{(z^2+4)(z^2+9)} \right| |\mathrm{d}z|$$

Now we take R sufficiently large. By triangle inequality, we obtain

$$\left| \int_{C_R} \frac{z+1}{(z^2+4)(z^2+9)}\, \mathrm{d}z \right| \leqslant \int_{C_R} \frac{|z|+1}{(|z|^2-4)(|z|^2-9)}\, |\mathrm{d}z| = \pi \frac{R\,(R+1)}{(R^2-4)(R^2-9)}.$$

Clearly when we take $R \to \infty$, the right-hand side above converges to 0. Therefore it holds

$$\lim_{R\to\infty} \int_{C_R} \frac{z+1}{(z^2+4)(z^2+9)}\, \mathrm{d}z = 0.$$

One should pay attention to the arguments in Example 8. It will be used in the following when we evaluate improper integrals of real-valued functions.

Sect. 9 Antiderivative and Independence on Path From (1.35) we know that if there is an analytic function f on Ω so that $g = f'$, then the antiderivative function of $g(z(t))z'(t)$ (i.e. $f'(z(t))z'(t)$) can be easily found and equals to $f(z(t))$. Here $z(t)$ is a parametrization of a differentiable curve in Ω. This motivates us a question. For what kind of function g can we have an analytic function f on Ω so that $g = f'$.

Definition 1.33. *Suppose g is a complex-valued function defined on an open set Ω. If there is an analytic function f on Ω so that $g = f'$, then we call f an antiderivative of g.*

Using this definition, we can rephrase our previous question as follows:

Q. Given a complex-valued function g on the open set Ω, can we find a criterion to determine if g admits an antiderivative on Ω?

Sect. 9.1 To answer this question, let us assume firstly that $g = f'$ on Ω for some analytic function f. Let l denote a contour in Ω. Moreover we assume

$$l = \bigcup_{j=1}^{N} l_j,$$

where l_j is a differentiable Jordan curve. Then we have

$$\int_l g(z)\,\mathrm{d}z = \int_l f'(z)\,\mathrm{d}z = \sum_{j=1}^{N} \int_{l_j} f'(z)\,\mathrm{d}z.$$

Suppose that $z_j(t)$ with $t \in [a_j, b_j]$ is a parametrization of l_j. Then the last equalities can be reduced to

$$\int_l g(z)\,\mathrm{d}z = \sum_{j=1}^{N} \int_{a_j}^{b_j} f'(z_j(t))\,z_j'(t)\,\mathrm{d}t.$$

Applying (1.35) to the last equality yields

$$\int_l g(z)\,\mathrm{d}z = \sum_{j=1}^{N} f(z_j(t))\Big|_{a_j}^{b_j} = \sum_{j=1}^{N} \big(f(z_j(b_j)) - f(z_j(a_j))\big).$$

Since for any j, it holds $z_j(b_j) = z_{j+1}(a_{j+1})$. It then follows from the last equality that

$$\int_l g(z)\,\mathrm{d}z = \sum_{j=1}^{N} f(z_j(t))\Big|_{a_j}^{b_j} = \sum_{j=1}^{N} \big(f(z_j(b_j)) - f(z_j(a_j))\big) = f(z_N(b_N)) - f(z_1(a_1)). \tag{1.42}$$

Notice that $z_N(b_N)$ is the end-point of the contour l and $z_1(a_1)$ is the initial point of the contour l. Therefore the last equality tells us that if g has an antiderivative on Ω, then for any contour l, the integral

$$\int_l g(z)\,\mathrm{d}z$$

depends only on the initial point and end-point of l. It is independent of the path itself. This property is called independence of path.

Sect. 9.2 Particularly if contour l is closed in the sense that $z_N(b_N)) = z_1(a_1)$, then by (1.42), it always holds

$$\int_l g(z)\,\mathrm{d}z = 0. \tag{1.43}$$

In other words (1.43) always holds, provided that g has antiderivative on Ω and l is a closed contour in Ω. Since for any rectangle in Ω, its edges form a closed contour in Ω, it also holds

$$\int_{\text{boundary of a rectangle}} g(z)\,\mathrm{d}z = 0, \qquad \text{for any rectangle contained in } \Omega. \tag{1.44}$$

Notice that boundary of a rectangle in (1.44) is a contour which is counter-clockwisely connected or clockwisely connected.

Sect. 9.3 Now we suppose that g is a complex-valued function satisfying (1.43) and Ω is a domain set. Fix a $z_0 \in \Omega$ as a reference point. For any z in Ω, since Ω is domain set, we can find a contour l_1 in Ω starting from z_0 and ending at z. By this contour l_1 we calculate

$$\int_{l_1} g(w)\,\mathrm{d}w. \tag{1.45}$$

Let l_2 be another contour starting from z_0 and ending at z. Then of course

$$l := l_1 \bigcup -l_2$$

forms a closed contour in Ω. Here $-l_2$ has the same set of points as l_2 but with different direction as l_2. Using (1.43), Definition 1.29 and Remark 1.27, we have

$$\int_l g(z)\,\mathrm{d}z = \int_{l_1} g(z)\,\mathrm{d}z + \int_{-l_2} g(z)\,\mathrm{d}z = \int_{l_1} g(z)\,\mathrm{d}z - \int_{l_2} g(z)\,\mathrm{d}z = 0.$$

Equivalently it holds

$$\int_{l_1} g(z)\,\mathrm{d}z = \int_{l_2} g(z)\,\mathrm{d}z.$$

This equality implies that once z_0 is fixed and z is fixed, the value of (1.45) is in fact independent of the path l_1. For any contour l starting from z_0 and ending at z, the integral

$$\int_l g(w)\,\mathrm{d}w$$

should be identical. Therefore we obtain a complex-valued function

$$f(z) = \int_{l_z} g(w)\,\mathrm{d}w, \tag{1.46}$$

where l_z denotes any contour contained in Ω, starting from z_0 and ending at z. We now need to show the analyticity of f in Ω. In what follows we always assume that g is continuous in Ω. To prove that $f(z)$ is analytic, we need to prove the existence and continuity of the first-order derivatives of f. Moreover we also need to show that the real and imaginary parts of f satisfy Cauchy-Riemann equation.

Fix an arbitrary z in Ω and let h_1, h_2 be two real numbers sufficiently small. The four points z, $z+h_1$, $z+ih_2$ and $z+\big(h_1+ih_2\big)$ form a rectangle whose four vertices are exactly the four points z, $z+h_1$, $z+ih_2$ and $z+\big(h_1+ih_2\big)$. Let L be a contour starting from z_0 and ending at $z+\big(h_1+ih_2\big)$. Let l_1 be the contour which starts from z_0 and move along L to $z+\big(h_1+ih_2\big)$, then move horizontally from $z+\big(h_1+ih_2\big)$ to $z+ih_2$, and then move vertically from $z+ih_2$ to z. Similarly we let l_2 be the contour which starts from z_0 and move along L to $z+\big(h_1+ih_2\big)$, then move vertically from $z+\big(h_1+ih_2\big)$ to $z+h_1$, and then move horizontally from $z+h_1$ to z. Of course l_1 and l_2 are two contours starting from z_0 and ending at z. By (1.46), it holds

$$f(z+h_1) = \int_L g(w)\,\mathrm{d}w + \int_{\left[z+\left(h_1+ih_2\right),z+h_1\right]} g(w)\,\mathrm{d}w.$$

Here $[w_1, w_2]$ denotes the segment starting from the point w_1 and ending at point w_2. Also by (1.46), we have

$$f(z) = \int_{l_2} g(w)\,\mathrm{d}w = \int_L g(w)\,\mathrm{d}w + \int_{\left[z+\left(h_1+ih_2\right),z+h_1\right]} g(w)\,\mathrm{d}w + \int_{[z+h_1,z]} g(w)\,\mathrm{d}w.$$

By the last two equalities, it follows

$$f(z+h_1) - f(z) = -\int_{[z+h_1,z]} g(w)\,\mathrm{d}w = \int_{[z,z+h_1]} g(w)\,\mathrm{d}w.$$

Therefore we obtain

$$\frac{f(z+h_1)-f(z)}{h_1} = \frac{1}{h_1}\int_{[z,z+h_1]} g(w)\,\mathrm{d}w \longrightarrow g(z), \quad \text{as } h_1 \to 0. \tag{1.47}$$

In the last convergence above, we have used mean value theorem and the continuity of g at z. Similarly (1.46) gives

$$f(z+ih_2) = \int_L g(w)\,\mathrm{d}w + \int_{\left[z+\left(h_1+ih_2\right),z+ih_2\right]} g(w)\,\mathrm{d}w.$$

Also by (1.46), we have

$$f(z) = \int_{l_1} g(w)\,\mathrm{d}w = \int_L g(w)\,\mathrm{d}w + \int_{\left[z+\left(h_1+ih_2\right),z+ih_2\right]} g(w)\,\mathrm{d}w + \int_{[z+ih_2,z]} g(w)\,\mathrm{d}w.$$

By the last two equalities, it follows

$$f(z+ih_2) - f(z) = -\int_{[z+ih_2,z]} g(w)\,\mathrm{d}w = \int_{[z,z+ih_2]} g(w)\,\mathrm{d}w.$$

Therefore we obtain

$$\frac{f(z+ih_2)-f(z)}{h_2} = \frac{1}{h_2}\int_{[z,z+ih_2]} g(w)\,\mathrm{d}w \longrightarrow i\,g(z), \quad \text{as } h_2 \to 0. \tag{1.48}$$

Here we also have used mean value theorem and the continuity of g at z. We now let $f = u + iv$ and let $z = x + iy$. Then it holds

$$\begin{aligned}\frac{f(z+h_1)-f(z)}{h_1} &= \frac{u(x+h_1,y)+iv(x+h_1,y)-u(x,y)-iv(x,y)}{h_1} \\ &= \frac{u(x+h_1,y)-u(x,y)}{h_1} + i\frac{v(x+h_1,y)-v(x,y)}{h_1}\end{aligned}$$

Applying (1.47) to the last equality and using the definition of partial derivatives, we obtain

$$\partial_x u\Big|_{(x,y)} + i\,\partial_x v\Big|_{(x,y)} = g(z). \tag{1.49}$$

Similarly we have

$$\begin{aligned}\frac{f(z+ih_2)-f(z)}{h_2} &= \frac{u(x,y+h_2)+iv(x,y+h_2)-u(x,y)-iv(x,y)}{h_2} \\ &= \frac{u(x,y+h_2)-u(x,y)}{h_2} + i\frac{v(x,y+h_2)-v(x,y)}{h_2}\end{aligned}$$

Applying (1.48) to the last equality and using the definition of partial derivatives, we obtain

$$\partial_y u\Big|_{(x,y)} + i\,\partial_y v\Big|_{(x,y)} = i\,g(z). \tag{1.50}$$

By (1.49)-(1.50), we have existence and continuity of the first-order partial derivatives of u and v. Here the continuity comes from the assumption that g is continuous on Ω. Moreover (1.49)-(1.50) also imply the satisfaction of Cauchy-Riemann equation by u and v. By Theorem 1.21, f defined in (1.46) is analytic throughout Ω. Moreover by (1.49) and Theorem 1.21, $f' = g$. In other word f defined in (1.46) is an antiderivative of g.

We now summarize all the arguments in this section as follows:

Theorem 1.34. *Suppose that g is a continuous function on the domain set Ω. Then g admits an antiderivative on Ω if and only if g satisfies the independent-of-path property.*

Some examples are followed.

Example 1. $f(z) = e^{\pi z}$. l is a contour starting from i and ending at $i/2$. Since $F(z) = e^{\pi z}/\pi$ is an antiderivative of f, it holds

$$\int_l f(z)\,\mathrm{d}z = \frac{e^{\pi z}}{\pi}\bigg|_i^{i/2} = \frac{1+i}{\pi}.$$

Example 2. The function $f(z) = 1/z^2$ is defined in the domain $\mathbb{C}\setminus\{0\}$. On this domain, $F(z) = -1/z$ is an antiderivative of f. Therefore it holds

$$\int_{\mathrm{Cir}(0;1)} f(z)\,\mathrm{d}z = 0.$$

Example 3. Let $f(z) = 1/z$. C_1 is the right-half

$$z(\theta) = e^{i\theta}, \qquad \theta \in [-\pi/2, \pi/2]$$

of the circle $\mathrm{Cir}(0;1)$. Clearly $F(z) = \mathrm{Log}z$ is an antiderivative of $f(z)$. Therefore it holds

$$\int_{C_1} \frac{1}{z}\,\mathrm{d}z = \mathrm{Log}z\bigg|_{-i}^{i} = \pi i. \tag{1.51}$$

Next let C_2 denote the left-half

$$z(\theta) = e^{i\theta}, \qquad \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$$

of $\mathrm{Cir}(0;1)$. Clearly $F(z) = \log z$ whose branch is given by $\arg z \in [0, 2\pi)$ is an antiderivative of f. It then holds

$$\int_{C_2} \frac{1}{z}\,\mathrm{d}z = \log z\bigg|_{i}^{-i} = \pi i. \tag{1.52}$$

Notice that $C_1 \bigcup C_2$ gives us the whole circle $\mathrm{Cir}(0;1)$. However by (1.51)-(1.52) we know that

$$\int_{C_1\bigcup C_2} \frac{1}{z}\,\mathrm{d}z = \int_{C_1} \frac{1}{z}\,\mathrm{d}z + \int_{C_2} \frac{1}{z}\,\mathrm{d}z = 2\pi i \neq 0.$$

It then follows that on $\mathbb{C}\setminus\{0\}$, $1/z$ does not satisfy the independent-of-path property. Hence by Theorem 1.34, $1/z$ does not have antiderivative on $\mathbb{C}\setminus\{0\}$. This is of course true since antiderivative of $1/z$ must be one of $\log z$. But $\log z$ is not analytic on the branch cut. It is in fact even not continuous on the branch cut. This argument shows that $\log z$ is an antiderivative of $1/z$ off the branch cut of $\log z$. If we have a closed contour l intersecting with the branch cut of $\log z$, we can not simply claim

$$\int_l \frac{1}{z}\,\mathrm{d}z = 0.$$

For C_1 and C_2 in this example we can apply Theorem 1.34. The reason is because the branch cut of $\log z$ used in these two cases have no intersection with C_1 and C_2, respectively.

Example 4. Suppose that

$$f(z) = \exp\left(\frac{1}{2}\log z\right) = \sqrt{|z|}\,e^{i\theta/2}, \qquad |z| > 0,\ \theta \in [0, 2\pi).$$

Let C_1 is any contour from $z = -3$ to $z = 3$ that, except for its end points, lies above the x-axis. We know that

$$\left(z^{3/2}\right)' = \frac{3}{2}z^{1/2}. \tag{1.53}$$

But in order to avoid intersection between C_1 and branch cut of $\log z$ used in the definition of power functions, we choose $\log z$ with branch defined on $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$. Now the branch cut is on the negative imaginary line, which has no intersection with C_1. Surely we have (1.53). Here one should notice that the branch is from $-\pi/2$ to $3\pi/2$. When we restrict on the upper-half plane, the argument should run from 0 to π. Therefore on the upper-half plane, $z^{1/2}$ defined in this branch equals to $\sqrt{|z|}\, e^{i\theta/2}$ with $\theta \in (0, \pi)$. It matches the restriction of f on C_1. It holds

$$f(z) = \left(\frac{2}{3} z^{3/2}\right)', \qquad \text{on the upper-half plane.}$$

Here $z^{3/2}$ is evaluated on the branch $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$. Therefore we get

$$\int_{C_1} f(z)\,\mathrm{d}z = \frac{2}{3} z^{3/2}\Big|_{-3}^{3} = 2\sqrt{3}\,(1+i)\,.$$

Sect. 10. Integration of analytic functions on closed loop In this section we assume f is an analytic function on $\overline{\Omega}$ where Ω is a domain set. Let $\partial\Omega$ be the boundary of Ω which is counter-clockwisely oriented. We are interested in the integral

$$\int_{\partial\Omega} f(z)\,\mathrm{d}z.$$

Sect. 10.1. Integral on boundary of rectangle To make geometry of Ω as simple as possible, we assume here Ω is a rectangle. Without loss of generality in the following argument we can assume Ω is a square with length of each edge equaling to a. The argument in the following can be easily generalized to rectangle case. In the current situation, we denote by Ω_0 the square Ω and let l_0 be the boundary of the square Ω. l_0 is also counter-clockwisely oriented. Let

$$I_0 = \int_{l_0} f(z)\,\mathrm{d}z \tag{1.54}$$

We now use middle points of each edge of Ω_0 to separate Ω_0 into four identical sub-squares (see Fig. 1). These four squares are denoted by $\Omega_{0,j}$ with $j = 1, ..., 4$. Their associated boundaries are denoted by $l_{0,j}$ with $j = 1, ..., 4$. Here $l_{0,j}$ is also counter-clockwisely oriented. It can be shown that

$$I_0 = \int_{l_{0,1}} f(z)\,\mathrm{d}z + \int_{l_{0,2}} f(z)\,\mathrm{d}z + \int_{l_{0,3}} f(z)\,\mathrm{d}z + \int_{l_{0,4}} f(z)\,\mathrm{d}z. \tag{1.55}$$

If for all $j = 1, ..., 4$, we have

$$\left|\int_{l_{0,j}} f(z)\,\mathrm{d}z\right| < \frac{1}{4}\,|\,I_0\,|, \tag{1.56}$$

then by triangle inequality we obtain from (1.55)-(1.56) that

$$\begin{aligned} |\,I_0\,| &\leqslant \left|\int_{l_{0,1}} f(z)\,\mathrm{d}z\right| + \left|\int_{l_{0,2}} f(z)\,\mathrm{d}z\right| + \left|\int_{l_{0,3}} f(z)\,\mathrm{d}z\right| + \left|\int_{l_{0,4}} f(z)\,\mathrm{d}z\right| \\ &< \frac{1}{4}\,|\,I_0\,| + \frac{1}{4}\,|\,I_0\,| + \frac{1}{4}\,|\,I_0\,| + \frac{1}{4}\,|\,I_0\,| \;=\; |\,I_0\,|. \end{aligned}$$

Hence it follows $|\,I_0\,| < |\,I_0\,|$. But this is impossible. In other words we must have one j in $\{1, ..., 4\}$ so that

$$\left|\int_{l_{0,j}} f(z)\,\mathrm{d}z\right| \geqslant \frac{1}{4}\,|\,I_0\,|. \tag{1.57}$$

Denote this $l_{0,j}$ by l_1. The square enclosed by l_1 is denoted by Ω_1. Moreover we let

$$I_1 = \int_{l_1} f(z)\,\mathrm{d}z.$$

(1.57) immediately implies

$$|\,I_1\,| \;\geqslant\; \frac{1}{4}\,|\,I_0\,|.$$

We can apply the above arguments to the square Ω_1 and obtain l_2 and its enclosed square Ω_2 so that $\Omega_2 \subset \Omega_1$ and

$$|\,I_2\,| \;\geqslant\; \frac{1}{4}\,|\,I_1\,|, \qquad \text{where } I_2 := \int_{l_2} f(z)\,\mathrm{d}z. \tag{1.58}$$

Repeating the above arguments inductively we have a sequence of l_n and a sequence of squares Ω_n with $\partial\Omega_n = l_n$ so that

$$\Omega_{n+1} \subset \Omega_n. \tag{1.59}$$

Moreover if we define

$$I_n := \int_{l_n} f(z)\,\mathrm{d}z,$$

then we also have

$$|\,I_n\,| \;\geqslant\; \frac{1}{4}\,|\,I_{n-1}\,|\,. \tag{1.60}$$

(1.60) shows that

$$|\,I_0\,| \leqslant 4\,|\,I_1\,| \leqslant 4^2\,|\,I_2\,| \leqslant ... \leqslant 4^n\,|\,I_n\,| \tag{1.61}$$

Notice (1.59), the sequence of squares are shrinking to a point, denoted by $z_0 \in \overline{\Omega}$. Since f is analytic at z_0, it must holds

$$\lim_{z\,\to\,z_0} \left|\, \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0) \,\right| = 0. \tag{1.62}$$

As for I_n, it satisfies

$$I_n = \int_{l_n} f(z)\,\mathrm{d}z = \int_{l_n} f(z) - f(z_0) - f'(z_0)(z-z_0)\,\mathrm{d}z = \int_{l_n} \left[\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right]\Big(z-z_0\Big)\,\mathrm{d}z.$$

Therefore we get

$$|\,I_n\,| \;\leqslant\; \max_{z\,\in\,l_n} \left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| \int_{l_n} \big|z-z_0\big|\,\big|\mathrm{d}z\big|. \tag{1.63}$$

Notice that for any n, z_0 is in $\overline{\Omega}_n$. For any z on l_n, $|z-z_0|$ is bounded from above by the largest distance of two points in $\overline{\Omega}_n$. This largest distance is achieved by the length of the diagonal of Ω_n. Since the length of each edge of Ω_n is $2^{-n}a$. It holds

$$|z-z_0| \;\leqslant\; \sqrt{2}\,2^{-n}\,a, \qquad \text{for any } z \in l_n.$$

Applying this estimate to (1.63) yields

$$|\,I_n\,| \;\leqslant\; \max_{z\,\in\,l_n} \left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| \sqrt{2}\,2^{-n}\,a \int_{l_n} \big|\mathrm{d}z\big| = \sqrt{2}\,a^2\,4^{1-n} \max_{z\,\in\,l_n} \left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right|.$$

Applying this estimate to the right-hand side of (1.61) and utilizing (1.62), we get

$$|\,I_0\,| \;\leqslant\; 4\sqrt{2}\,a^2 \max_{z\,\in\,l_n} \left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| \longrightarrow 0, \qquad \text{as } n \to \infty.$$

Therefore we have

Theorem 1.35 (Cauchy-Gousat)**.** *Suppose f is an analytic function on the closure of an rectangle. Let l be the boundary of this rectangle counter-clockwisely oriented. Then it holds*

$$\int_l f(z)\,\mathrm{d}z = 0.$$

Sect. 10.2. Integral on any closed contour in a rectangle Cauchy-Gousat theorem is a building block for our generalization. But in Theorem 1.35, the geometry of the contour is too restrictive. In fact it is a boundary of a rectangle. Now we assume f is an analytic function on a closed rectangle Ω and generalize Theorem 1.35 to

$$\int_l f(z)\,\mathrm{d}z = 0, \qquad \text{for any closed contour } l \text{ contained in } \Omega. \tag{1.64}$$

From Sect. 9, (1.64) is true if we know that f has an antiderivative on Ω. Now we need to construct an antiderivative of f. Let z_0 be the center of the rectangle Ω. For any $z \in \Omega$, we can connect z_0 and z by the two ways shown in Fig. 2. Cauchy-Gousat theorem implies that

$$\int_{L_1} f(w)\,\mathrm{d}w = \int_{L_2} f(w)\,\mathrm{d}w.$$

Therefore we can also define

$$F(z) := \int_{L_1} f(w)\,\mathrm{d}w = \int_{L_2} f(w)\,\mathrm{d}w.$$

We then can apply the same arguments as the first part of Sect. 9.3 to show that $F(z)$ is analytic and satisfies

$$F'(z) = f(z).$$

Therefore (1.64) follows by Theorem 1.34. That is

Theorem 1.36. *Suppose f is an analytic function on the closure of an rectangle. Then (1.64) holds.*

Sect. 10.3. Simply connected domain Now we assume Ω is a simply connected domain. Let $l_0 = \partial\Omega$ be counter-clockwisely oriented. We can deform l_0 a little bit to l_1, where l_1 is also counter-clockwisely oriented (see Fig.3). Moreover we can let the distance between l_0 and l_1 to be very small. Then we slice the strip between l_0 and l_1 into many small pieces. Each piece is so tiny that it can be contained in a rectangle on which f is analytic. Therefore by Theorem 1.36, the contour integral on boundary of each small piece must be 0. Summing contour integrals on boundaries of all small pieces, we have

$$\int_{l_0} f(z)\,\mathrm{d}z + \int_{-l_1} f(z)\,\mathrm{d}z = 0.$$

Here the integrals on common edges are also cancelled out. The above equality show that

$$\int_{l_0} f(z)\,\mathrm{d}z = \int_{l_1} f(z)\,\mathrm{d}z.$$

Since Ω is simply connected without holes, we can deform l_0 to a point, denoted by z_0, in Ω by a sequence of contours l_0, l_1, ..., l_n ... For each l_j and l_{j+1}, we can make their distance to be very small. By the previous arguments, it follows

$$\int_{l_0} f(z)\,\mathrm{d}z = \int_{l_n} f(z)\,\mathrm{d}z, \qquad \text{for any } n. \tag{1.65}$$

Since f is analytic on $\overline{\Omega}$, f must be bounded. We get

$$\left|\int_{l_n} f(z)\,\mathrm{d}z\right| \leqslant \int_{l_n} |f(z)||\mathrm{d}z| \leqslant \max_{z\in\overline{\Omega}} |f(z)|\,\mathrm{length}(l_n) \longrightarrow 0, \qquad \text{as } n \to \infty.$$

By this limit and (1.65), we get

$$\int_{l_0} f(z)\,\mathrm{d}z = 0.$$

That is

Theorem 1.37. *Suppose f is an analytic function on the closure of a simply connected domain Ω. Then*

$$\int_{\partial\Omega} f(z)\,\mathrm{d}z = 0.$$

Sect. 10.4. Multiple connected domain For domain with holes we can separate it into finitely many simply connected domains, as shown in Fig.4. Then Theorem 1.37 can be applied on the boundary of each sub-domain. Noticing the direction induced on the boundaries of holes, we can easily get

Theorem 1.38. *Suppose f is an analytic function on the closure of the multiple connected domain Ω. Let l_0 be the exterior boundary and l_j with $j = 1, ..., N$ be the interior boundaries. If l_0, ..., l_N are all counter-clockwisely oriented, then it holds*

$$\int_{l_0} f(z)\,\mathrm{d}z = \sum_{j=1}^{N} \int_{l_j} f(z)\,\mathrm{d}z.$$

Sect. 11. Cauchy Integral Formula. Fixing a domain set Ω, we assume that f is analytic on the closure of Ω. In terms of f, we define

$$g(z) := \frac{f(z) - f(z_0)}{z - z_0}, \qquad \text{for all } z \in \overline{\Omega} \setminus \left\{z_0\right\}.$$

Here z_0 is an arbitrary point in Ω. Let $\epsilon > 0$ be small enough so that the disk $D(z_0; \epsilon) \subset \Omega$. It is clear that g is analytic on the closure of $\Omega \setminus D(z_0; \epsilon)$. Applying Theorem 1.38 to g, we get

$$\int_{\partial\Omega} g(z)\,\mathrm{d}z = \int_{\mathrm{Cir}(z_0;\epsilon)} g(z)\,\mathrm{d}z.$$

Here $\partial\Omega$ and $\mathrm{Cir}(z_0; \epsilon)$ are all counter-clockwisel oriented. Plugging the representation of g into the above equality yields

$$\left| \int_{\partial\Omega} \frac{f(z) - f(z_0)}{z - z_0}\,\mathrm{d}z \right| = \left| \int_{\mathrm{Cir}(z_0;\epsilon)} \frac{f(z) - f(z_0)}{z - z_0}\,\mathrm{d}z \right| \leqslant \int_{\mathrm{Cir}(z_0;\epsilon)} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| |\mathrm{d}z|.$$

Since f is analytic on the closure of Ω, $g(z)$ is uniformly bounded on $\overline{\Omega}$. The last estimate gives us

$$\begin{aligned} \left| \int_{\partial\Omega} \frac{f(z) - f(z_0)}{z - z_0}\,\mathrm{d}z \right| \leqslant \int_{\mathrm{Cir}(z_0;\epsilon)} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| |\mathrm{d}z| &\leqslant \max_{z \in \overline{\Omega}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| \int_{\mathrm{Cir}(z_0;\epsilon)} |\mathrm{d}z| \\ &= 2\pi\epsilon \max_{z \in \overline{\Omega}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right|. \end{aligned}$$

Since ϵ can be arbitrarily small, therefore we can take $\epsilon \to 0$ on the most-right-hand side above and get

$$\int_{\partial\Omega} \frac{f(z) - f(z_0)}{z - z_0}\,\mathrm{d}z = 0.$$

Equivalently it follows

$$f(z_0) \int_{\partial\Omega} \frac{1}{z - z_0}\,\mathrm{d}z = \int_{\partial\Omega} \frac{f(z)}{z - z_0}\,\mathrm{d}z. \tag{1.66}$$

Noticing that $\frac{1}{z-z_0}$ is also analytic on the closure of $\Omega \setminus D(z_0;\epsilon)$, we can apply Theorem 1.38 one more time for the function $\frac{1}{z-z_0}$ to obtain

$$\int_{\partial\Omega} \frac{1}{z-z_0}\,\mathrm{d}z = \int_{\mathrm{Cir}(z_0;\epsilon)} \frac{1}{z-z_0}\,\mathrm{d}z = 2\pi i.$$

The last equality above uses Example 3 in Sect. 9.3. Moreover by applying the last equality into (1.66) yields

$$f(z_0) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(z)}{z-z_0}\,\mathrm{d}z. \tag{1.67}$$

This formulae is the famous Cauchy's integral formula. Several applications of (1.67) can be carried out as follows.

App 1. The integral on the right-hand side of (1.67) is on the boundary of Ω. In other words we only need information of f on $\partial\Omega$, then the right-hand side of (1.67) can be evaluated. We donot need any information of f inside Ω. However z_0 is an arbitrary point in Ω. The left-hand side of (1.67) tells us the value of f at z_0. Therefore (1.67) indeed gives us a representation formulae for the value of f at any point in Ω in terms of the integral on $\partial\Omega$. That is to say that the value of f is uniquely determined by its values on $\partial\Omega$.

App 2. If we have an analytic function f, then we can rewrite (1.67) as follows:

$$\int_{\partial\Omega} \frac{f(z)}{z-z_0}\,\mathrm{d}z = 2\pi i f(z_0). \tag{1.68}$$

Therefore for any integral of the type given on the left-hand side of (1.68), we can simply evaluated it by $2\pi i$ times the value of f at the given location z_0.

Example 1. Let C be the counter-clockwisely oriented unit circle $\mathrm{Cir}(0;1)$. Let $f(z) = \frac{\cos z}{z^2+9}$. Since f is analytic on the closure of $D(0;1)$, it holds by (1.68) that

$$\int_C \frac{\cos z}{z(z^2+9)}\,\mathrm{d}z = \int_C \frac{f(z)}{z-0}\,\mathrm{d}z = 2\pi i f(0) = \frac{2\pi i}{9}.$$

App 3. Fixing an arbitrary z_0 in Ω and taking h a complex number with small modulus, we can have $z_0+h \in \Omega$. Therefore by (1.67), we get

$$f(z_0+h) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(z)}{z-z_0-h}\,\mathrm{d}z$$

Subtracting (1.67) from the above equality yields

$$f(z_0+h) - f(z_0) = \frac{1}{2\pi i}\int_{\partial\Omega} f(z)\left(\frac{1}{z-z_0-h} - \frac{1}{z-z_0}\right)\mathrm{d}z = \frac{1}{2\pi i}\int_{\partial\Omega} f(z)\,\frac{(z-z_0)-(z-z_0-h)}{(z-z_0-h)(z-z_0)}\,\mathrm{d}z.$$

Equivalently it follows

$$\frac{f(z_0+h)-f(z_0)}{h} = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(z)}{(z-z_0-h)(z-z_0)}\,\mathrm{d}z.$$

By using this equality, we obtain

$$\begin{aligned}\frac{f(z_0+h)-f(z_0)}{h} - \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(z)}{(z-z_0)^2}\,\mathrm{d}z &= \frac{1}{2\pi i}\int_{\partial\Omega} f(z)\left[\frac{1}{(z-z_0-h)(z-z_0)} - \frac{1}{(z-z_0)^2}\right]\mathrm{d}z \\ &= \frac{h}{2\pi i}\int_{\partial\Omega} \frac{f(z)}{(z-z_0-h)(z-z_0)^2}\,\mathrm{d}z. \end{aligned} \tag{1.69}$$

Since $z \in \partial\Omega$ and $z_0 \in \Omega$, then it must hold

$$\big|z - z_0\big| \geqslant \min_{w \in \partial\Omega} \big|w - z_0\big| > 0, \qquad \text{for any } z \in \partial\Omega \,. \tag{1.70}$$

Moreover we can take h so that $|h| \leqslant \dfrac{1}{2}\min_{w \in \partial\Omega}\big|w - z_0\big|$. Then by triangle inequality, it holds

$$|z - z_0 - h| \geqslant |z - z_0| - |h| \geqslant \min_{w \in \partial\Omega}\big|w - z_0\big| - \frac{1}{2}\min_{w \in \partial\Omega}\big|w - z_0\big| = \frac{1}{2}\min_{w \in \partial\Omega}\big|w - z_0\big|, \quad \text{for any } z \in \partial\Omega. \tag{1.71}$$

Applying (1.70)-(1.71) to (1.69), we get

$$\begin{aligned}
\left|\frac{f(z_0+h) - f(z_0)}{h} - \frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(z)}{(z-z_0)^2}\,\mathrm{d}z\right| \;&\leqslant\; \frac{|h|}{2\pi}\int_{\partial\Omega}\frac{|f(z)|}{|z-z_0-h||z-z_0|^2}\,|\,\mathrm{d}z\,| \\
&\leqslant\; \frac{|h|}{\pi}\,\frac{\max_{z \in \partial\Omega}|f(z)|}{\Big(\min_{w \in \partial\Omega}\big|w - z_0\big|\Big)^3}\,\mathrm{length}(\partial\Omega).
\end{aligned}$$

Obviously if we take $h \to 0$, the right-hand side above converges to 0. In other words by the definition of complex derivative, it holds

$$f'(z_0) = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(z)}{(z-z_0)^2}\,\mathrm{d}z, \qquad \text{for any } z_0 \in \Omega. \tag{1.72}$$

(1.72) is the Cauchy integral formulae for the derivative of f.

Inductively we assume that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{\partial\Omega}\frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z, \qquad \text{for any } z_0 \in \Omega, \tag{1.73}$$

here $f^{(n)}$ denotes the n-th order derivative of f, then we can repeat the above arguments and get

$$\begin{aligned}
f^{(n)}(z_0+h) - f^{(n)}(z_0) \;&=\; \frac{n!}{2\pi i}\int_{\partial\Omega} f(z)\left[\frac{1}{(z-z_0-h)^{n+1}} - \frac{1}{(z-z_0)^{n+1}}\right]\mathrm{d}z \\
&=\; \frac{n!}{2\pi i}\int_{\partial\Omega} f(z)\,\frac{(z-z_0)^{n+1} - (z-z_0-h)^{n+1}}{(z-z_0-h)^{n+1}(z-z_0)^{n+1}}\mathrm{d}z.
\end{aligned}$$

Equivalently it follows

$$\frac{f^{(n)}(z_0+h) - f^{(n)}(z_0)}{h} = \frac{n!}{2\pi i}\int_{\partial\Omega} f(z)\,\frac{(z-z_0)^{n+1} - (z-z_0-h)^{n+1}}{h}\,\frac{1}{(z-z_0-h)^{n+1}(z-z_0)^{n+1}}\mathrm{d}z. \tag{1.74}$$

When we take $h \to 0$, it holds

$$\frac{1}{(z-z_0-h)^{n+1}(z-z_0)^{n+1}} \longrightarrow \frac{1}{(z-z_0)^{2n+2}}, \qquad \text{for any } z \in \partial\Omega. \tag{1.75}$$

Moreover as $h \to 0$, it also has

$$\frac{(z-z_0)^{n+1} - (z-z_0-h)^{n+1}}{h} \longrightarrow (n+1)(z-z_0)^n. \tag{1.76}$$

Applying (1.75)-(1.76) to (1.74) we get

$$\frac{f^{(n)}(z_0+h) - f^{(n)}(z_0)}{h} \longrightarrow \frac{(n+1)!}{2\pi i}\int_{\partial\Omega}\frac{f(z)}{(z-z_0)^{n+2}}\,\mathrm{d}z, \qquad \text{as } h \to 0.$$

Equivalently we get

$$f^{(n+1)}(z_0) = \frac{(n+1)!}{2\pi i}\int_{\partial\Omega} \frac{f(z)}{(z-z_0)^{n+2}}\,\mathrm{d}z, \qquad \text{for any } z_0 \in \Omega,$$

The above arguments indeed imply that not just the first-order derivative of f, f has any n-th order derivative, which can be represented in terms of (1.73). (1.73) is also called generalized Cauchy integral formulae. One has to be noticed that. Initially we only assume the analyticity of f without any information on the higher-order derivatives of f. But by using Cauchy integral formulae (1.67), we can show (1.73) holds for any natural number n. That is f, once analytic, then it must have all the higher order derivatives automatically.

App 4. If we have an analytic function f, then we can rewrite (1.73) as follows:

$$\int_{\partial\Omega} \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z = \frac{2\pi i}{n!} f^{(n)}(z_0). \tag{1.77}$$

Therefore for any integral of the type given on the left-hand side of (1.77), we can simply evaluated it by the right-hand side of (1.77).

Example 2. Let C be the counter-clockwisely oriented unit circle $\mathrm{Cir}(0;1)$ and let $f(z) = e^{2z}$. Then it holds

$$\int_C \frac{e^{2z}}{z^4}\,\mathrm{d}z = \frac{\pi i}{3} f^{(3)}(0) = \frac{8\pi i}{3}.$$

Sect. 12. Liouville's theorem and the fundamental theorem of algebra. In this section we assume f is an entire function and satisfies

$$\left| f(z) \right| \leqslant M, \qquad \text{for any } z \in \mathbb{C}. \tag{1.78}$$

Here $M > 0$ is a constant. Fix an arbitrary $R > 0$ and let C_R be the circle centered at 0 with radius R. Moreover we let C_R to be counter-clockwisely oriented. For any $z_0 \in \mathbb{C}$, we can take $R > |z_0|$ so that $z_0 \in D(0;R)$. Here $D(0;R)$ is the disk with center 0 and radius R. Then by Cauchy integral formula, we have

$$f'(z_0) = \frac{1}{2\pi i}\int_{C_R} \frac{f(z)}{(z-z_0)^2}\,\mathrm{d}z, \qquad \text{for any } R > |z_0|.$$

By this equality and (1.78), it follows

$$\left| f'(z_0) \right| \leqslant \frac{1}{2\pi}\int_{C_R} \frac{|f(z)|}{|z-z_0|^2}\,|\mathrm{d}z| \leqslant \frac{M}{2\pi}\int_{C_R} \frac{1}{|z-z_0|^2}\,|\mathrm{d}z|. \tag{1.79}$$

Since it has $|z - z_0| \geqslant |z| - |z_0| = R - |z_0|$ for any $z \in C_R$, we can keep estimating the right-hand side of (1.79) as follows:

$$\left| f'(z_0) \right| \leqslant \frac{M}{2\pi}\frac{1}{(R-|z_0|)^2}\int_{C_R} |\mathrm{d}z| = \frac{MR}{(R-|z_0|)^2} \longrightarrow 0, \qquad \text{as } R \to \infty.$$

Therefore we have $f'(z_0) = 0$ for any $z_0 \in \mathbb{C}$. In other words f must be a constant. This is the Liouville's theorem stated as below:

Theorem 1.39. *If f is an entire function satisfying (1.78) for some $M > 0$, then f must be a constant.*

An application of Liouvilles's theorem is a proof for the fundamental theorem of algebra. Let

$$P(z) = a_0 + \ldots + a_{n-1}z^{n-1} + a_n z^n = q(z) + a_n z^n, \quad \text{where } a_n \neq 0.$$

By triangle inequality we easily have

$$|P(z)| \geqslant |a_n||z|^n - |q(z)|. \tag{1.80}$$

Since the highest order of polynomial q can not exceed $n-1$, it holds

$$\lim_{z\to\infty} \frac{|q(z)|}{|a_n||z|^n} = 0.$$

There then has a $R > 0$, so that for any z with $|z| > R$, it satisfies

$$|q(z)| \leqslant \frac{1}{2}|a_n||z|^n.$$

Applying the above inequality to the right-hand side of (1.80) yields

$$|P(z)| \geqslant \frac{1}{2}|a_n||z|^n > \frac{1}{2}|a_n|R^n, \qquad \text{for any } z \text{ with } |z| > R. \tag{1.81}$$

Equivalently it has

$$\frac{1}{|P(z)|} \leqslant \frac{2}{|a_n|R^n}, \qquad \text{for any } z \text{ with } |z| > R. \tag{1.82}$$

If $p(z)$ has no root on $\mathbb{C}$, then $\frac{1}{p(z)}$ is an entire function on $\mathbb{C}$. By (1.82), we must have

$$\frac{1}{|P(z)|} \leqslant \max\left\{ \max_{z\in D(0;R)} \frac{1}{|p(z)|}, \frac{2}{|a_n|R^n} \right\}, \qquad \text{for any } z \in \mathbb{C}.$$

Hence by applying Liouville's theorem to $\frac{1}{p(z)}$, $\frac{1}{p(z)}$ must be a constant function, which is impossible. That is

Theorem 1.40. *Any non-constant polynomial must have at least one root on $\mathbb{C}$.*

Sect. 13. Maximum Modulus Theorem In this section we use Cauchy integral formula to study the so-called Maximum Modulus Theorem. Firstly we assume $\Omega = D(z_0; R)$. f is an analytic function on the closure of Ω. Then by Cauchy integral formula, it holds

$$f(z_0) = \frac{1}{2\pi i}\int_{\mathrm{Cir}(z_0;R')} \frac{f(z)}{z - z_0}\,\mathrm{d}z, \qquad \text{for any } R' \in \big(0, R\big].$$

Here $\mathrm{Cir}(z_0; R')$ is counter-clockwisely oriented. If the maximum value of $|f(z)|$ in the closure of Ω is achieved at z_0, then we get from the above equality that

$$|f(z_0)| \leqslant \frac{1}{2\pi}\int_{\mathrm{Cir}(z_0;R')} \frac{|f(z)|}{|z-z_0|}|\mathrm{d}z| \leqslant \frac{1}{2\pi}\int_{\mathrm{Cir}(z_0;R')} \frac{|f(z_0)|}{|z-z_0|}|\mathrm{d}z| = \frac{|f(z_0)|}{2\pi R'} \times 2\pi R' = |f(z_0)|.$$

In fact the second inequality above should be an equality. Rewriting the second inequality above yields

$$\frac{1}{2\pi}\int_{\mathrm{Cir}(z_0;R')} \frac{|f(z_0)| - |f(z)|}{|z - z_0|}|\mathrm{d}z| = 0.$$

However the integrand above is a non-negative function. This implies $|f(z)| = |f(z_0)|$ for any $z \in \mathrm{Cir}(z_0; R')$. Since R' is an any number in $\big(0, R\big]$, we get $|f(z)| = |f(z_0)|$ for any $z \in D(z_0; R)$. By Example 3 in Sect. 7, f must be a constant function.

Now we assume Ω to be an arbitrary domain set in $\mathbb{C}$. And let f be analytic on $\overline{\Omega}$. If there is $z_0 \in \Omega$ on which $|f(z)|$ takes its maximum value over $\overline{\Omega}$, then by the above arguments, for some disk $D(z_0; r) \subset \Omega$, f should be a constant function. Now we let z_1 be an arbitrary point in Ω. Since Ω is path-connected, we can find a differentiable curve l connecting z_0 and z_1. Meanwhile l is contained in Ω. Suppose l is parameterized by $z(t)$ with $t \in [a, b]$. Then we know for some $\epsilon > 0$, $f(z(t))$ should be a constant on $[a, a+\epsilon]$. Here we assume $z(a) = z_0$, $z(b) = z_1$. $\epsilon > 0$ is sufficiently small so that $z(t)$ with $t \in [a, a+\epsilon]$ is contained in $D(z_0; r)$. Denote by ϵ_{max} the largest number in $(0, b-a]$ so that $f(z(t))$ is a constant in $[a, a+\epsilon_{max}]$. By continuity we have $f(z(a+\epsilon_{max})) = f(z_0)$. If $\epsilon_{max} < b - a$, then we can find another radius r' so that $f(z)$ is a constant function on $D(z(a+\epsilon_{max}); r')$. Therefore we have another $\epsilon' > 0$ suitably small so that $f(z(t))$ is a constant in $[a+\epsilon_{max}, a+\epsilon_{max}+\epsilon']$. This is a contradiction to the definition of ϵ_{max}. Hence it holds $\epsilon_{max} = b - a$. Equivalently it holds $f(z_0) = f(z_1)$. Since z_1 is an arbitrary point in Ω, then we get $f(z) = f(z_0)$ for any $z \in \Omega$.

The above arguments imply that if the maximum value of $|f(z)|$ over $\overline{\Omega}$ is achieved by an interior point $z_0 \in \Omega$, then f must be a constant function. In other words if f is analytic on $\overline{\Omega}$ and is not a constant function, then the largest modulus of $f(z)$ can only be achieved by its boundary point. That is

Theorem 1.41. *If a function f is analytic and not constant on the closure of a domain set Ω, then $|f(z)|$ has no maximum value in Ω. That is, there is no point $z_0 \in \Omega$ such that $|f(z)| \leqslant |f(z_0)|$ for all points $z \in \overline{\Omega}$.*

Example 1. Fundamental theorem of algebra. Theorem 1.41 can also be applied to prove Theorem 1.40. Suppose $p(z)$ is a non-constant polynomial. If $p(z)$ has no root in $\mathbb{C}$, then for any $r > 0$, $\dfrac{1}{p(z)}$ must be analytic on the closure of $D(0; r)$. By Theorem 1.41, it holds

$$\frac{1}{|p(z)|} \leqslant \max_{z \in \mathrm{Cir}(0;r)} \frac{1}{|p(z)|}, \qquad \text{for any } z \in D(0; r). \tag{1.83}$$

Let R be the same radius as in (1.81) and take $r > R$. Then by the first inequality in (1.81), it holds

$$|P(z)| \geqslant \frac{1}{2}|a_n| r^n, \qquad \text{for any } z \in \mathrm{Cir}(0; r).$$

Applying this estimate to the right-hand side of (1.83) yields

$$\frac{1}{|p(z)|} \leqslant \frac{2}{|a_n| r^n}, \qquad \text{for any } z \in D(0; r).$$

Taking $r \to \infty$, we get $\dfrac{1}{p(z)} = 0$ for any $z \in \mathbb{C}$. This is impossible. The proof is done.

Example 2. Consider the function $f(z) = (z+1)^2$ defined on the closed triangle region R with vertices at the points $z = 0$, $z = 2$ and $z = i$. A simple geometric argument can be used to locate points in R at which the modulus $|f(z)|$ has its maximum and minimum values. The arguments is based on the interpretation of $|f(z)|$ as the square of the distance d between -1 and any point $z \in R$:

$$d^2 = |f(z)| = |z - (-1)|^2.$$

As one can see, the maximum and minimum values of d, and therefore $|f(z)|$, occur at boundary points, namely $z = 2$ and $z = 0$, respectively.

Sect. 14. Taylor's and Laurent's series In this section we study two important series related to analytic functions. In the following arguments, $D(z_0; R_0)$ is the open disk with center z_0 and radius R_0. $A(z_0; R_1, R_2)$ is the open annulus with center z_0, interior radius R_1 and exterior radius R_2. Firstly let us consider Taylor series of an analytic function f on $\overline{D(z_0; R_0)}$.

Sect. 14.1. Taylor's series Suppose that f is analytic on $\overline{D(z_0; R_0)}$. Then for any $z \in D(z_0; R_0)$, we can apply Cauchy integral formula to get

$$f(z) = \frac{1}{2\pi i}\int_{\mathrm{Cir}(z_0;R_0)} \frac{f(w)}{w-z}\,\mathrm{d}w = \frac{1}{2\pi i}\int_{\mathrm{Cir}(z_0;R_0)} \frac{f(w)}{(w-z_0)-(z-z_0)}\,\mathrm{d}w. \tag{1.84}$$

As for the denominator, we rewrite

$$(w-z_0)-(z-z_0) = (w-z_0)\left\{1-\frac{z-z_0}{w-z_0}\right\}.$$

Applying this equality to the right-hand side of (1.84) yields

$$f(z) = \frac{1}{2\pi i}\int_{\mathrm{Cir}(z_0;R_0)} \frac{f(w)}{w-z_0}\,\frac{1}{1-\dfrac{z-z_0}{w-z_0}}\,\mathrm{d}w, \qquad \text{for any } z \in D(z_0;R_0). \tag{1.85}$$

Since $z \in D(z_0;R_0)$ and $w \in \mathrm{Cir}(z_0;R_0)$, we have $|z-z_0| < R_0$ and $|w-z_0| = R_0$. This implies $\left|\dfrac{z-z_0}{w-z_0}\right| < 1$. By geometric series, it follows

$$\frac{1}{1-\dfrac{z-z_0}{w-z_0}} = \sum_{j=0}^{\infty}\left[\frac{z-z_0}{w-z_0}\right]^j.$$

Applying this equality to (1.85) and fixing an arbitrary natural number N, we obtain

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{\mathrm{Cir}(z_0;R_0)} \frac{f(w)}{w-z_0}\sum_{j=0}^{\infty}\left[\frac{z-z_0}{w-z_0}\right]^j\,\mathrm{d}w \\
&= \frac{1}{2\pi i}\int_{\mathrm{Cir}(z_0;R_0)} \frac{f(w)}{w-z_0}\sum_{j=0}^{N}\left[\frac{z-z_0}{w-z_0}\right]^j\,\mathrm{d}w + \frac{1}{2\pi i}\int_{\mathrm{Cir}(z_0;R_0)} \frac{f(w)}{w-z_0}\sum_{j=N+1}^{\infty}\left[\frac{z-z_0}{w-z_0}\right]^j\,\mathrm{d}w \\
&= \sum_{j=0}^{N}\frac{1}{2\pi i}\int_{\mathrm{Cir}(z_0;R_0)} \frac{f(w)}{w-z_0}\left[\frac{z-z_0}{w-z_0}\right]^j\,\mathrm{d}w + \frac{1}{2\pi i}\int_{\mathrm{Cir}(z_0;R_0)} \frac{f(w)}{w-z_0}\sum_{j=N+1}^{\infty}\left[\frac{z-z_0}{w-z_0}\right]^j\,\mathrm{d}w \\
&= \sum_{j=0}^{N}\left(\frac{1}{2\pi i}\int_{\mathrm{Cir}(z_0;R_0)} \frac{f(w)}{(w-z_0)^{j+1}}\,\mathrm{d}w\right)(z-z_0)^j + \frac{1}{2\pi i}\int_{\mathrm{Cir}(z_0;R_0)} \frac{f(w)}{w-z_0}\sum_{j=N+1}^{\infty}\left[\frac{z-z_0}{w-z_0}\right]^j\,\mathrm{d}w,
\end{aligned}$$

for any $z \in D(z_0;R_0)$.

In terms of the general Cauchy integral formula (see (1.77)), it holds

$$\frac{1}{2\pi i}\int_{\mathrm{Cir}(z_0;R_0)} \frac{f(w)}{(w-z_0)^{j+1}}\,\mathrm{d}w = \frac{f^{(j)}(z_0)}{j!}.$$

By the last two equalities, we get

$$f(z) = \sum_{j=0}^{N}\frac{f^{(j)}(z_0)}{j!}(z-z_0)^j + \frac{1}{2\pi i}\int_{\mathrm{Cir}(z_0;R_0)} \frac{f(w)}{w-z_0}\sum_{j=N+1}^{\infty}\left[\frac{z-z_0}{w-z_0}\right]^j\,\mathrm{d}w, \quad z \in D(z_0;R_0). \tag{1.86}$$

Rewriting the last equality implies

$$f(z) - \sum_{j=0}^{N}\frac{f^{(j)}(z_0)}{j!}(z-z_0)^j = \frac{1}{2\pi i}\int_{\mathrm{Cir}(z_0;R_0)} \frac{f(w)}{w-z_0}\sum_{j=N+1}^{\infty}\left[\frac{z-z_0}{w-z_0}\right]^j\,\mathrm{d}w, \quad z \in D(z_0;R_0).$$

Hence it follows

$$\begin{aligned}
\left|f(z) - \sum_{j=0}^{N}\frac{f^{(j)}(z_0)}{j!}(z-z_0)^j\right| &= \frac{1}{2\pi}\left|\int_{\mathrm{Cir}(z_0;R_0)} \frac{f(w)}{w-z_0}\sum_{j=N+1}^{\infty}\left[\frac{z-z_0}{w-z_0}\right]^j\,\mathrm{d}w\right| \\
&\leqslant \frac{1}{2\pi}\int_{\mathrm{Cir}(z_0;R_0)} \frac{|f(w)|}{|w-z_0|}\left|\sum_{j=N+1}^{\infty}\left[\frac{z-z_0}{w-z_0}\right]^j\right|\,|\,\mathrm{d}w\,|
\end{aligned}$$

Suppose that $|f(w)| \leqslant M$ for all $w \in \overline{D(z_0; R_0)}$ and some $M > 0$. Then by triangle inequality, the last estimate is reduced to

$$
\begin{aligned}
\left| f(z) - \sum_{j=0}^{N} \frac{f^{(j)}(z_0)}{j!}(z-z_0)^j \right| &\leqslant \frac{M}{2\pi R_0} \int_{\mathrm{Cir}(z_0;R_0)} \sum_{j=N+1}^{\infty} \left| \frac{z-z_0}{w-z_0} \right|^j |\,\mathrm{d}w\,| \\
&= \frac{M}{2\pi R_0} \int_{\mathrm{Cir}(z_0;R_0)} \sum_{j=N+1}^{\infty} \left(\frac{|\,z-z_0\,|}{R_0} \right)^j |\,\mathrm{d}w\,| \\
&= M \sum_{j=N+1}^{\infty} \left(\frac{|\,z-z_0\,|}{R_0} \right)^j = M \frac{\dfrac{|z-z_0|^{N+1}}{R_0^{N+1}}}{1 - \dfrac{|z-z_0|}{R_0}}.
\end{aligned}
$$

Since $|z-z_0| < R_0$, we have $\dfrac{|z-z_0|^{N+1}}{R_0^{N+1}} \longrightarrow 0$, as $N \to \infty$. We then can take $N \to \infty$ in the last estimate and obtain

$$
f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z-z_0)^j, \qquad z \in D(z_0; R_0). \tag{1.87}
$$

(1.87) is the famous Taylor series expansion of an analytic function on $\overline{D(z_0; R_0)}$.

Example 1. Let $f(z) = \dfrac{1}{1-z}$. On $|\,z\,| < 1$, it holds

$$
f^{(j)}(z) = \frac{j!}{(1-z)^{j+1}}.
$$

Therefore on $|\,z\,| < 1$, we have

$$
f(z) = \frac{1}{1-z} = \sum_{j=0}^{\infty} z^j.
$$

Example 2. Let $f(z) = e^z$. On $\mathbb{C}$, it holds

$$
f^{(j)}(z) = e^z.
$$

Therefore on $\mathbb{C}$, we have

$$
f(z) = e^z = \sum_{j=0}^{\infty} \frac{1}{j!} z^j.
$$

Example 3. Let $f(z) = \sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$. On $\mathbb{C}$, it holds

$$
f^{(j)}(z) = \frac{i^j e^{iz} - (-i)^j e^{-iz}}{2i}.
$$

Therefore on $\mathbb{C}$, we have

$$
f(z) = \sin z = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{i^j - (-i)^j}{2i} z^j = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} i^{2k} z^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}.
$$

Sect. 14.2. Laurent's series Now we assume f is analytic on $\overline{A(z_0; R_1, R_2)}$. For any $z \in A(z_0; R_1, R_2)$, we can take $\epsilon > 0$ sufficiently small so that $\overline{D(z;\epsilon)} \subset A(z_0; R_1, R_2)$. Obviously $A(z_0; R_1, R_2) \setminus \overline{D(z;\epsilon)}$ is a multiple connected domain with $\mathrm{Cir}(z_0; R_2)$ being its exterior boundary. Moreover $\mathrm{Cir}(z_0; R_1)$ and $\mathrm{Cir}(z; \epsilon)$ are interior boundaries of $A(z_0; R_1, R_2) \setminus \overline{D(z;\epsilon)}$. By multiple connected version of Cauchy's theorem, we have

$$\int_{\mathrm{Cir}(z_0;R_2)} \frac{f(w)}{w-z}\,\mathrm{d}w = \int_{\mathrm{Cir}(z_0;R_1)} \frac{f(w)}{w-z}\,\mathrm{d}w + \int_{\mathrm{Cir}(z;\epsilon)} \frac{f(w)}{w-z}\,\mathrm{d}w. \tag{1.88}$$

Here $\mathrm{Cir}(z_0; R_1)$, $\mathrm{Cir}(z_0; R_2)$ and $\mathrm{Cir}(z; \epsilon)$ are all counter-clockwisely oriented. By (1.68), it follows

$$\int_{\mathrm{Cir}(z;\epsilon)} \frac{f(w)}{w-z}\,\mathrm{d}w = 2\pi i\, f(z).$$

Applying this equality to the right-hand side of (1.88) yields

$$f(z) = \frac{1}{2\pi i}\int_{\mathrm{Cir}(z;\epsilon)} \frac{f(w)}{w-z}\,\mathrm{d}w = \frac{1}{2\pi i}\int_{\mathrm{Cir}(z_0;R_2)} \frac{f(w)}{w-z}\,\mathrm{d}w - \frac{1}{2\pi i}\int_{\mathrm{Cir}(z_0;R_1)} \frac{f(w)}{w-z}\,\mathrm{d}w. \tag{1.89}$$

In the following we deal with the two terms on the most-right-hand side of (1.89). By the same arguments as in Sect.14.1 for the Taylor series, it holds

$$\frac{1}{2\pi i}\int_{\mathrm{Cir}(z_0;R_2)} \frac{f(w)}{w-z}\,\mathrm{d}w = \sum_{j=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\mathrm{Cir}(z_0;R_2)} \frac{f(w)}{(w-z_0)^{j+1}}\,\mathrm{d}w\right)(z-z_0)^j. \tag{1.90}$$

Now we consider the last term in (1.89). In fact we also have

$$w - z = (w - z_0) - (z - z_0).$$

But in this case $z \in A(z_0; R_1, R_2)$ implies that $|z - z_0| > R_1 = |w - z_0|$, for any $w \in \mathrm{Cir}(z_0; R_1)$. Different from Taylor's series, we take $-(z - z_0)$ in front in the last equality and get

$$\frac{1}{w-z} = \frac{1}{-(z-z_0)}\frac{1}{1-\dfrac{w-z_0}{z-z_0}} = -\frac{1}{z-z_0}\sum_{k=0}^{\infty}\left(\frac{w-z_0}{z-z_0}\right)^k, \qquad z \in A(z_0; R_1, R_2).$$

By this equality we have, for any natural number N, that

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\mathrm{Cir}(z_0;R_1)} \frac{f(w)}{w-z}\,\mathrm{d}w &= -\frac{1}{2\pi i}\frac{1}{z-z_0}\int_{\mathrm{Cir}(z_0;R_1)} f(w)\sum_{k=0}^{\infty}\left(\frac{w-z_0}{z-z_0}\right)^k \mathrm{d}w \\
&= -\frac{1}{2\pi i}\frac{1}{z-z_0}\int_{\mathrm{Cir}(z_0;R_1)} f(w)\sum_{k=0}^{N}\left(\frac{w-z_0}{z-z_0}\right)^k \mathrm{d}w \\
&- \frac{1}{2\pi i}\frac{1}{z-z_0}\int_{\mathrm{Cir}(z_0;R_1)} f(w)\sum_{k=N+1}^{\infty}\left(\frac{w-z_0}{z-z_0}\right)^k \mathrm{d}w \\
&= -\frac{1}{2\pi i}\frac{1}{z-z_0}\sum_{k=0}^{N}\int_{\mathrm{Cir}(z_0;R_1)} f(w)\left(\frac{w-z_0}{z-z_0}\right)^k \mathrm{d}w \\
&- \frac{1}{2\pi i}\frac{1}{z-z_0}\int_{\mathrm{Cir}(z_0;R_1)} f(w)\sum_{k=N+1}^{\infty}\left(\frac{w-z_0}{z-z_0}\right)^k \mathrm{d}w.
\end{aligned}$$

Same as for the Taylor's series case, it follows

$$\left| \frac{1}{2\pi i}\int_{\mathrm{Cir}(z_0;R_1)} \frac{f(w)}{w-z}\,\mathrm{d}w + \frac{1}{2\pi i}\frac{1}{z-z_0}\sum_{k=0}^{N}\int_{\mathrm{Cir}(z_0;R_1)} f(w)\left(\frac{w-z_0}{z-z_0}\right)^k \mathrm{d}w \right|$$

$$\leqslant \frac{M}{2\pi|z-z_0|}\int_{\mathrm{Cir}(z_0;R_1)}\sum_{j=N+1}^{\infty}\left|\frac{w-z_0}{z-z_0}\right|^j |\,\mathrm{d}w\,| = \frac{M}{2\pi|z-z_0|}\int_{\mathrm{Cir}(z_0;R_1)}\sum_{j=N+1}^{\infty}\left(\frac{R_1}{|z-z_0|}\right)^j |\,\mathrm{d}w\,|$$

$$= M\sum_{j=N+2}^{\infty}\left(\frac{R_1}{|z-z_0|}\right)^j = M\frac{\dfrac{R_1^{N+2}}{|z-z_0|^{N+2}}}{1-\dfrac{R_1}{|z-z_0|}} \longrightarrow 0, \qquad \text{as } N\to\infty.$$

Therefore we get

$$\frac{1}{2\pi i}\int_{\mathrm{Cir}(z_0;R_1)} \frac{f(w)}{w-z}\,\mathrm{d}w = -\frac{1}{2\pi i}\sum_{k=0}^{\infty}(z-z_0)^{-(k+1)}\int_{\mathrm{Cir}(z_0;R_1)} f(w)(w-z_0)^k \mathrm{d}w. \tag{1.91}$$

Changing variable by letting $j=-(k+1)$ in (1.91), we have $j=-1,-2,....$ Here one needs to know that the index k in (1.91) runs from 0, 1, ... Now we can reduce (1.91) to

$$\frac{1}{2\pi i}\int_{\mathrm{Cir}(z_0;R_1)} \frac{f(w)}{w-z}\,\mathrm{d}w = -\frac{1}{2\pi i}\sum_{j=-1}^{-\infty}(z-z_0)^j\int_{\mathrm{Cir}(z_0;R_1)} \frac{f(w)}{(w-z_0)^{j+1}}\mathrm{d}w. \tag{1.92}$$

Applying (1.90) and (1.92) to the most-right-hand side of (1.89), we obtain

$$f(z)=\sum_{j=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\mathrm{Cir}(z_0;R_2)} \frac{f(w)}{(w-z_0)^{j+1}}\,\mathrm{d}w\right)(z-z_0)^j + \frac{1}{2\pi i}\sum_{j=-1}^{-\infty}\left(\int_{\mathrm{Cir}(z_0;R_1)} \frac{f(w)}{(w-z_0)^{j+1}}\mathrm{d}w\right)(z-z_0)^j.$$

In fact for any $R\in[R_1,R_2]$, it follows

$$\int_{\mathrm{Cir}(z_0;R_1)} \frac{f(w)}{(w-z_0)^{j+1}}\mathrm{d}w = \int_{\mathrm{Cir}(z_0;R_2)} \frac{f(w)}{(w-z_0)^{j+1}}\mathrm{d}w = \int_{\mathrm{Cir}(z_0;R)} \frac{f(w)}{(w-z_0)^{j+1}}\mathrm{d}w, \quad j\in\mathbb{Z}.$$

The last two equalities then imply

$$f(z)=\sum_{j=-\infty}^{\infty}\left(\frac{1}{2\pi i}\int_{\mathrm{Cir}(z_0;R)} \frac{f(w)}{(w-z_0)^{j+1}}\,\mathrm{d}w\right)(z-z_0)^j, \qquad \text{for any } z\in A(z_0;R_1,R_2). \tag{1.93}$$

(1.93) is the famous Laurent's series for analytic functions on annulus.

Remark 1.42. *If f is also analytic on $D(z_0;R_2)$, then f admits a Taylor series on $D(z_0;R_2)$. One can show that in this case Taylor series of f on $D(z_0;R_2)$ and Laurent series of f on any $A(z_0;r,R_2)$ agree with each other. Here $r\in(0,R_2)$. In fact all coefficients of negative indices in the Laurent series of f equal to* 0 *by Cauchy theorem.*

Remark 1.43. *If f is analytic on $\overline{D(z_0;R_0)}$ and can be represented by*

$$f(z)=\sum_{j=0}^{\infty}a_j(z-z_0)^j, \tag{1.94}$$

then a_j must be Taylor coefficient for any $j=0,1,....$ Samely if f is analytic on $\overline{a(z_0;R_1,R_2)}$ and can be represented by

$$f(z)=\sum_{j=-\infty}^{\infty}a_j(z-z_0)^j,$$

then a_j must be Laurent coefficient for any $j \in \mathbb{Z}$. We only consider the Taylor series case. As for Laurent series, the proof is similar. Let k be a fixed natural number. By (1.94), it holds

$$\frac{f(z)}{(z-z_0)^{k+1}} = \sum_{j=0}^{k-1} a_j (z-z_0)^{j-(k+1)} + \frac{a_k}{z-z_0} + \sum_{j=k+1}^{N} a_j (z-z_0)^{j-(k+1)} + \sum_{j=N+1}^{\infty} a_j (z-z_0)^{j-(k+1)}.$$

Now for any $R \in (0, R_0)$, we integrate the above equality over $\mathrm{Cir}(z_0; R)$ and get

$$\begin{aligned}\int_{\mathrm{Cir}(z_0;R)} \frac{f(z)}{(z-z_0)^{k+1}} &= \sum_{j=0}^{k-1} a_j \int_{\mathrm{Cir}(z_0;R)} (z-z_0)^{j-(k+1)} \\ &+ \int_{\mathrm{Cir}(z_0;R)} \frac{a_k}{z-z_0} \\ &+ \int_{\mathrm{Cir}(z_0;R)} \sum_{j=k+1}^{N} a_j (z-z_0)^{j-(k+1)} + \int_{\mathrm{Cir}(z_0;R)} \sum_{j=N+1}^{\infty} a_j (z-z_0)^{j-(k+1)}.\end{aligned}$$

By Theorem 1.34, the first and third integral on the right-hand side above equals to 0, in that integrands in these integrations all have antiderivative functions. Therefore the last equality is reduced to

$$\frac{2\pi i}{k!} f^{(k)}(z_0) = 2\pi i\, a_k + \int_{\mathrm{Cir}(z_0;R)} \sum_{j=N+1}^{\infty} a_j (z-z_0)^{j-(k+1)}, \qquad \textit{for any natural number } N. \tag{1.95}$$

Here we have used (1.77). Let z^* be a point on $\mathrm{Cir}(z_0; R_0)$. Since the series

$$\sum_{j=0}^{\infty} a_j (z^* - z_0)^j$$

converges, there is a constant $M > 0$ so that

$$|a_j|\,|z^* - z_0|^j \leqslant M, \qquad \textit{for any } j = 1, \ldots.$$

Therefore for any $z \in \mathrm{Cir}(z_0; R)$, it satisfies

$$\left|a_j (z-z_0)^j\right| = \left| a_j (z^*-z_0)^j \left[\frac{z-z_0}{z^*-z_0}\right]^j \right| = \left|a_j (z^*-z_0)^j\right| \left| \left[\frac{z-z_0}{z^*-z_0}\right]^j \right| \leqslant M \left[\frac{R}{R_0}\right]^j, \ \textit{for any } j = 1, \ldots.$$

By the last estimate, triangle inequality and geometric series, it follows

$$\begin{aligned}\left| \int_{\mathrm{Cir}(z_0;R)} \sum_{j=N+1}^{\infty} a_j (z-z_0)^{j-(k+1)}\, \mathrm{d}z \right| &\leqslant \int_{\mathrm{Cir}(z_0;R)} \sum_{j=N+1}^{\infty} \left| a_j (z-z_0)^{j-(k+1)} \right| |\mathrm{d}z| \\ &\leqslant 2\pi R^{-k} M \sum_{j=N+1}^{\infty} \left[\frac{R}{R_0}\right]^j \longrightarrow 0, \ \textit{as } N \to \infty.\end{aligned}$$

By this limit we can take $N \to \infty$ in (1.95) and get

$$a_k = \frac{1}{k!} f^{(k)}(z_0), \qquad \textit{for any } k = 1, \ldots.$$

The above equality still holds for $k = 0$. Therefore once an analytic function can be represented by series (1.94) in a $D(z_0; R_0)$, then this series must be Taylor series.

Example 4. Let $f(z) = \dfrac{1}{z(1+z^2)}$. This f is well-defined on $0 < |z| < 1$. Since we have

$$\frac{1}{1+z^2} = \sum_{j=0}^{\infty} (-1)^k z^{2k}, \qquad |z| < 1.$$

Therefore on $0 < |\,z\,| < 1$, we have

$$f(z) = \sum_{j=0}^{\infty} (-1)^k z^{2k-1}.$$

This is the Laurent series of f.

Example 5. Let $f(z) = \dfrac{z+1}{z-1}$. If $|\,z\,| < 1$, then

$$f(z) = -z\frac{1}{1-z} - \frac{1}{1-z} = -z\sum_{j=0}^{\infty} z^j - \sum_{j=0}^{\infty} z^j = -1 - 2\sum_{j=1}^{\infty} z^j.$$

This is the Taylor series of f on $|\,z\,| < 1$. If $|\,z\,| > 1$, then we have

$$f(z) = \frac{1+\frac{1}{z}}{1-\frac{1}{z}} = \left(1+\frac{1}{z}\right)\sum_{j=0}^{\infty}\frac{1}{z^j} = \sum_{j=0}^{\infty}\frac{1}{z^j} + \sum_{j=0}^{\infty}\frac{1}{z^{j+1}} = 1 + 2\sum_{j=1}^{\infty}\frac{1}{z^j}.$$

This is the Laurent series of f on $|\,z\,| > 1$.

Example 6. By Example 2, it holds

$$e^{1/z} = \sum_{j=0}^{\infty}\frac{1}{j!}z^{-j}.$$

This is the Laurent series expansion of $e^{1/z}$ with center 0. By comparing the coefficients with (1.93), we have

$$\frac{1}{2\pi i}\int_{\mathrm{Cir}(0;R)}\frac{e^{1/w}}{w^{1-j}}\,\mathrm{d}w = \frac{1}{j!}, \qquad j = 0, 1, \ldots.$$

In particular, it follows

$$\int_{\mathrm{Cir}(0;R)}\frac{e^{1/w}}{w^{1-j}}\,\mathrm{d}w = \frac{2\pi i}{j!}, \qquad j = 0, 1, \ldots.$$

Sect. 15. Isolated Singularities. In this section we assume that f is analytic on the punctured disk

$$\Big\{z \in \mathbb{C} : 0 < |z - z_0| \leqslant R_0\Big\}.$$

Clearly for any $0 < r_1 < r_2 \leqslant R_0$, f is analytic on the closure of $A(z_0; r_1, r_2)$. Therefore f can be expanded by the Laurent series as follows:

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j, \qquad \text{for any } z \in \Big\{z \in \mathbb{C} : 0 < |z - z_0| \leqslant R_0\Big\}. \tag{1.96}$$

There are three cases that might happen from the above Laurent series expansion. Case I. $a_j = 0$ for any $j \leqslant -1$; Case II. There is a natural number N_0 so that $a_j = 0$ for any $j \leqslant -N_0 - 1$. But $a_{-N_0} \neq 0$; Case III. There are infinitely many negative integers, denoted by j_1, j_2, ..., j_k, ... so that $a_{j_k} \neq 0$. We are going to study these three cases in this section.

Sect. 15.1. Removable Singularity. Firstly we consider case I. By the above assumption, it holds

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j, \qquad \text{for any } z \in \Big\{z \in \mathbb{C} : 0 < |z - z_0| \leqslant R_0\Big\}.$$

Initially the function f has no definition at z_0. But the series on the right-hand side above has definition at z_0. In fact if we plug $z = z_0$ into the series on the right-hand side above, we obtain

$$\sum_{j=0}^{\infty} a_j (z - z_0)^j = a_0, \qquad \text{at } z = z_0.$$

In other words by letting $f(z_0) = a_0$, we can extend the definition of f from the punctured disk $\Big\{z : 0 < |z - z_0| \leqslant R_0\Big\}$ to the whole closed disk $\Big\{z : |z - z_0| \leqslant R_0\Big\}$. In the following we still use f to denote this extended function of f. Now comes a question. Is this f analytic throughout the whole closed disk $\Big\{z : |z - z_0| \leqslant R_0\Big\}$? Here we only need to check the differentiability of f at the newly defined location z_0. By the definition of f at z_0, it holds

$$f(z) - f(z_0) = \sum_{j=0}^{\infty} a_j (z - z_0)^j - a_0 = \sum_{j=1}^{\infty} a_j (z - z_0)^j.$$

Therefore

$$\frac{f(z) - f(z_0)}{z - z_0} = \sum_{j=1}^{\infty} a_j (z - z_0)^{j-1} = a_1 + \sum_{j=2}^{N} a_j (z - z_0)^{j-1} + \sum_{j=N+1}^{\infty} a_j (z - z_0)^{j-1}. \tag{1.97}$$

Let z^* by a fixed point on $\mathrm{Cir}(z_0; R_0)$. The series

$$\sum_{j=0}^{\infty} a_j (z^* - z_0)^j$$

is convergent. One can find a constant $M > 0$ so that

$$\Big| a_j (z^* - z_0)^j \Big| \leqslant M, \qquad \text{for any } j = 0, \ldots.$$

By this upper bound, it holds

$$\begin{aligned}
\Big| \sum_{j=N+1}^{\infty} a_j (z - z_0)^{j-1} \Big| &= \frac{1}{|z^* - z_0|} \Big| \sum_{j=N+1}^{\infty} a_j (z^* - z_0)^j \left[\frac{z - z_0}{z^* - z_0} \right]^{j-1} \Big| \\
&\leqslant \frac{1}{|z^* - z_0|} \sum_{j=N+1}^{\infty} \Big| a_j (z^* - z_0)^j \Big| \Big| \left[\frac{z - z_0}{z^* - z_0} \right]^{j-1} \Big| \\
&\leqslant \frac{M}{|z^* - z_0|} \sum_{j=N+1}^{\infty} \Big| \frac{z - z_0}{z^* - z_0} \Big|^{j-1} = \frac{M}{R_0} \sum_{j=N+1}^{\infty} \left[\frac{|z - z_0|}{R_0} \right]^{j-1}.
\end{aligned}$$

Since we need to take $z \to z_0$, we can assume $|z - z_0| < R_0$. By geometric series, the above estimate can be reduced to

$$\Big| \sum_{j=N+1}^{\infty} a_j (z - z_0)^{j-1} \Big| \leqslant \frac{M}{R_0} \sum_{j=N+1}^{\infty} \left[\frac{|z - z_0|}{R_0} \right]^{j-1} = \frac{M}{R_0} \frac{\left[\dfrac{|z - z_0|}{R_0} \right]^N}{1 - \dfrac{|z - z_0|}{R_0}}.$$

Utilizing this estimate and (1.97), we get

$$\begin{aligned}
\Big| \frac{f(z) - f(z_0)}{z - z_0} - a_1 \Big| &\leqslant \Big| \sum_{j=2}^{N} a_j (z - z_0)^{j-1} \Big| + \Big| \sum_{j=N+1}^{\infty} a_j (z - z_0)^{j-1} \Big| \\
&\leqslant \sum_{j=2}^{N} \big| a_j (z - z_0)^{j-1} \big| + \frac{M}{R_0} \frac{\left[\dfrac{|z - z_0|}{R_0} \right]^N}{1 - \dfrac{|z - z_0|}{R_0}}.
\end{aligned}$$

By taking $z \to z_0$, it holds from the above estimate that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - a_1 \right| \longrightarrow 0, \qquad \text{as } z \to z_0.$$

In other word, the newly defined function f is derivable at z_0. Therefore f is analytic throughout the whole closed disk $\Big\{z : |z - z_0| \leqslant R_0\Big\}$. This comes the name of this type singularity. In fact you can see this singularity can be removed by redefining f at z_0 to be the a_0 in the Laurent series expansion of f.

Sect. 15.2. Poles. Now we consider Case II. As in the assumption of Case II, $f(z)$ in (1.96) can be written as

$$f(z) = \sum_{j=-N_0}^{\infty} a_j (z - z_0)^j, \qquad \text{where } a_{-N_0} \neq 0.$$

Taking $(z - z_0)^{-N_0}$ in front, we get

$$f(z) = \frac{\sum_{j=-N_0}^{\infty} a_j (z - z_0)^{j+N_0}}{(z - z_0)^{N_0}} = \frac{\sum_{k=0}^{\infty} a_{k-N_0} (z - z_0)^k}{(z - z_0)^{N_0}}.$$

We denote by $g(z)$ the function

$$g(z) = \sum_{k=0}^{\infty} a_{k-N_0} (z - z_0)^k.$$

Similarly as in Sect. 15.1, this function g must be analytic throughout the whole closed disk $\Big\{z : |z - z_0| \leqslant R_0\Big\}$. Moreover

$$g(z_0) = a_{-N_0} \neq 0. \tag{1.98}$$

From the above arguments, we know that f in this case can be represented by

$$f(z) = \frac{g(z)}{(z - z_0)^{N_0}},$$

where $g(z)$ is analytic throughout $\Big\{z : |z - z_0| \leqslant R_0\Big\}$ with $g(z_0) \neq 0$. By the above representation, it holds

$$|f(z)| = \frac{|g(z)|}{|z - z_0|^{N_0}} \longrightarrow \frac{|g(z_0)|}{0} = \infty, \qquad \text{as } z \to z_0.$$

In this case z_0 is called pole of f with order N_0.

Sect. 15.3. Essential Singularity. The singularity in Case III is called essential singularity. From the first two cases, in Case I, it holds

$$f(z) \longrightarrow c, \qquad \text{as } z \to z_0.$$

Here c is some constant. In case II, it holds

$$f(z) \longrightarrow \infty, \qquad \text{as } z \to z_0.$$

Therefore we can guess that in this case f neither converges to a finite constant c, nor diverges to ∞ as $z \to z_0$. In fact we can show

Proposition 1.44. *For any complex number c, there is a sequence $z_n \to z_0$ so that it holds*

$$\big|f(z_n) - c\big| \longrightarrow 0, \qquad \textit{as } n \to \infty.$$

To prove this result we need the following lemma.

Lemma 1.45. *If f is analytic on $\Big\{z \in \mathbb{C} : 0 < |z - z_0| \leqslant R_0\Big\}$ and*

$$|f| \leqslant M, \qquad on \Big\{z \in \mathbb{C} : 0 < |z - z_0| \leqslant R_0\Big\}, \tag{1.99}$$

for some constant $M > 0$, then z_0 is a removable singularity of f.

Proof. Suppose

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j, \qquad z \in \Big\{z \in \mathbb{C} : 0 < |z - z_0| \leqslant R_0\Big\}.$$

For any $j \leqslant -1$, it holds

$$\int_{\mathrm{Cir}(z_0;R_0)} \frac{f(w)}{(w - z_0)^{j+1}}\, \mathrm{d}w = \int_{\mathrm{Cir}(z_0;\epsilon)} \frac{f(w)}{(w - z_0)^{j+1}}\, \mathrm{d}w. \tag{1.100}$$

Here $\epsilon > 0$ is any small radius. The above equality holds by multiple connected version of Cauchy theorem. By (1.99), we have

$$\begin{aligned} \left| \int_{\mathrm{Cir}(z_0;\epsilon)} \frac{f(w)}{(w - z_0)^{j+1}}\, \mathrm{d}w \right| \quad &\leqslant \quad \int_{\mathrm{Cir}(z_0;\epsilon)} \frac{|\, f(w)\,|}{|w - z_0|^{j+1}}\, |\, \mathrm{d}w\,| \\ &\leqslant \quad 2\pi M \epsilon^{-j} \longrightarrow 0, \qquad \text{as } \epsilon \to 0. \end{aligned}$$

Applying this limit to (1.100) yields

$$\int_{\mathrm{Cir}(z_0;R_0)} \frac{f(w)}{(w - z_0)^{j+1}}\, \mathrm{d}w = 0, \qquad j = -1, -2, ...$$

Therefore the Laurent series $a_j = 0$ for any $j = -1, -2, ...$ Therefore f can be represented by

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j.$$

Similar argument as before implies that z_0 is a removable singularity of f. □

Now we prove Proposition 1.44.

Proof of Proposition 1.44. If on the contrary Proposition 1.44 fails to be true, then there is a complex number c, an $\epsilon_0 > 0$ and $r_0 \in (0, R_0)$ so that

$$|\, f(z) - c\,| \geqslant \epsilon_0, \qquad \text{for } z \in \Big\{z : 0 < |z - z_0| \leqslant r_0\Big\}. \tag{1.101}$$

Denote by g the function

$$g(z) = \frac{1}{f(z) - c}.$$

By (1.101), this g is analytic on $\Big\{z : 0 < |z - z_0| \leqslant r_0\Big\}$. Moreover it holds

$$|\, g(z)\,| \leqslant \frac{1}{\epsilon_0}, \qquad \text{for any } z \in \Big\{z : 0 < |z - z_0| \leqslant r_0\Big\}.$$

Applying Lemma 1.45 to this function g, we get $g(z)$ has a removable singularity at z_0. In other words

$$g(z) = \frac{1}{f(z) - c} = \sum_{j=0}^{\infty} b_j (z - z_0)^j, \qquad z \in \Big\{z : 0 < |z - z_0| \leqslant r_0\Big\}. \tag{1.102}$$

If $b_0 \neq 0$, then it holds

$$\lim_{z \to z_0} \sum_{j=0}^{\infty} b_j (z - z_0)^j = b_0 \neq 0.$$

By (1.102), it follows

$$\lim_{z \to z_0} f(z) = c + \frac{1}{b_0}.$$

This shows that $|f|$ must be uniformly bounded on $\Big\{z : 0 < |z - z_0| \leqslant R_0\Big\}$. Hence by Lemma 1.45, z_0 is a removable singularity of f. This is a contradiction since z_0 is assumed to be an essential singularity of f. If $b_0 = 0$ and there is a natural number N_0 such that $b_0 = ... = b_{N_0 - 1} = 0$, but $b_{N_0} \neq 0$, then it holds

$$\sum_{j=0}^{\infty} b_j (z - z_0)^j = (z - z_0)^{N_0} h(z).$$

Here $h(z)$ is an analytic function on $\Big\{z : |z - z_0| \leqslant r_0\Big\}$ with $h(z_0) \neq 0$. Still by (1.102) and the above equality, it follows

$$f(z) = c + \frac{h(z)^{-1}}{(z - z_0)^{N_0}}.$$

In this case z_0 is a pole of f with order N_0. This is still a contradiction to the assumption that z_0 is an essential singularity of f. The last case is when $b_j = 0$ for any $j = 0, 1, ...$ In this case we get from (1.102) that

$$\frac{1}{f(z) - c} = 0, \qquad z \in \Big\{z : 0 < |z - z_0| \leqslant r_0\Big\}.$$

This can happen if and only if $f = \infty$ on $\Big\{z : 0 < |z - z_0| \leqslant r_0\Big\}$. It is still impossible. Therefore statement in Proposition 1.44 must hold. □

Remark 1.46. *In fact we can not only approach each finite value c by f in a small neighborhood of z_0. If z_0 is an essential singularity of f, then it can assume every finite value, with one possible exception, an infinite number of times. This is the famous Picard's theorem. Its proof is omitted here. Interested readers may refer to Sec. 51 in Vol. III of the book: Theory of Functions of a Complex Variable by A. I. Markushevich.*

Now we use one example to illustrate Picard's theorem.

Example 1. Consider $f(z) = e^{1/z}$. Clearly $z = 0$ is the essential singularity of f. For any $z \in \Big\{z : 0 < |z| \leqslant 1\Big\}$, $f(z) \neq 0$. The value 0 is the only exceptional value which can not be taken by f in $\Big\{z : 0 < |z| \leqslant 1\Big\}$. In fact let $w_* = \rho_* e^{i\theta_*}$, where $\rho_* \neq 0$. We construct the equation

$$e^{1/z} = e^{\frac{x}{|z|^2}} e^{-i\frac{y}{|z|^2}} = \rho_* e^{i\theta_*}.$$

By the above equation it holds

$$e^{\frac{x}{|z|^2}} = \rho_* \qquad \text{and} \qquad e^{-i\frac{y}{|z|^2}} = e^{i\theta_*}. \tag{1.103}$$

The first equation in (1.103) tells us

$$\frac{x}{|z|^2} = \log \rho_*. \tag{1.104}$$

The second equation in (1.103) gives us

$$\frac{y}{|z|^2} = -\theta_* + 2n\pi, \qquad n \in \mathbb{Z}. \tag{1.105}$$

Connecting (1.104)-(1.105), we get

$$\frac{1}{|z|^2} = \big(\log \rho_*\big)^2 + \big(-\theta_* + 2n\pi\big)^2.$$

Utilizing this equality, we get from (1.104)-(1.105) that $z_n = x_n + i y_n$, where

$$x_n = \frac{\log \rho_*}{\big(\log \rho_*\big)^2 + \big(-\theta_* + 2n\pi\big)^2}, \qquad y_n = \frac{-\theta_* + 2n\pi}{\big(\log \rho_*\big)^2 + \big(-\theta_* + 2n\pi\big)^2}$$

are all solutions of (1.103). Equivalently at z_n, $f(z_n) = w_*$. Moreover one can easily check that as $n \to \infty$, $z_n \to 0$. i.e. w_* can be taken infinitely many times by f in any neighborhood of 0.

Sect. 16. Isolation of points in preimage. In this section we assume Ω is a bounded domain set. f is a non-constant analytic function throughout the closure of Ω. Let c be a value which can be taken by f. Now we consider the set of pre-image of c under the function f. That is

$$f^{-1}(c) := \Big\{ z \in \overline{\Omega} : f(z) = c \Big\}. \tag{1.106}$$

Suppose $z_0 \in f^{-1}(c)$. It holds $f(z_0) = c$. By Taylor expansion, we have

$$f(z) = f(z_0) + \sum_{j=N}^{\infty} a_j (z - z_0)^j = c + \sum_{j=N}^{\infty} a_j (z - z_0)^j, \qquad z \in D(z_0; \epsilon_0). \tag{1.107}$$

Here $\epsilon_0 > 0$ is small enough so that $D(z_0; \epsilon_0) \subset \Omega$. Moreover we assume $a_N \neq 0$ for some natural number N. Otherwise $f(z) = c$ for all points in $D(z_0; \epsilon_0)$, which implies that $f(z) = c$ for all $z \in \overline{\Omega}$. Without loss of generality, we still use N to denote the smallest index so that the corresponding Taylor coefficient of f is non-zero. By (1.107), it follows

$$f(z) - c = (z - z_0)^N h(z), \qquad \text{where } h(z) = a_N + a_{N+1}(z - z_0) + ... \tag{1.108}$$

Notice that $a_N \neq 0$. Therefore when ϵ_0 is small enough, it must hold

$$h(z) \neq 0, \qquad z \in D(z_0; \epsilon_0).$$

By this result and (1.108), we know that in $D(z_0; \epsilon_0)$, only at z_0 can we take the value c. In other words if f is not a constant function and f takes value c at another location z_1, then this location z_1 must keep ϵ_0 distance between z_0, at least. This property is the so-called isolation of points in preimage. With this result we can get the following analytic continuation result.

Proposition 1.47. *Let f and g be two analytic functions on $\overline{\Omega}$. If there is a sequence $z_n \in \overline{\Omega}$ so that $f(z_n) = g(z_n)$, then $f(z) = g(z)$ for all $z \in \Omega$.*

Proof. Let $h(z) = f(z) - g(z)$. By assumption $h(z_n) = 0$ for all n. If h does not identically equal to 0, then the locations where h take value 0 must be isolated. But $\overline{\Omega}$ is bounded. We can of course extract a subsequence of z_n so that the subsequence z_{n_k} converges to a point $z_* \in \overline{\Omega}$. By analyticity we also have $h(z_*) = 0$. But now z_* is not an isolated point at where $h = 0$. In any neighborhood of z_*, you can find a z_{n_k} for large k so that z_{n_k} lies in this neighborhood and $f(z_{n_k}) = 0$. □

An application of this proposition is the following reflection principle.

Theorem 1.48. *Suppose that* $\Omega = \Omega^+ \bigcup l \bigcup \Omega^-$, *where* Ω^+ *and* Ω^- *are symmetric with respect to the x-axis. Moreover we assume that* Ω^+ *is in the upper-half part of the complex plane* $\mathbb{C}$ *with* $\overline{\Omega^+} \bigcap \Big\{ \textit{real axis} \Big\} = l$. *Let f be an analytic function on* $\overline{\Omega}$. *Then*

$$\overline{f(z)} = f(\overline{z}), \qquad z \in \Omega \tag{1.109}$$

if and only if f is real-valued on l.

Proof. Let $z = x$ be a real number in l. Therefore $z = \overline{z}$. By (1.109), it holds $\overline{f(z)} = f(\overline{z}) = f(z)$. Therefore $f(z)$ must be real-valued on l. On the other hand let $g(z) = \overline{f(\overline{z})}$. It can be easily shown that g is also analytic on $\overline{\Omega}$. If f is real-valued on l, then it holds $g(z) = \overline{f(\overline{z})} = \overline{f(z)} = f(z)$ for all $z \in l$. By Proposition 1.47, it holds $g(z) = f(z)$ for all $z \in \Omega$. The proof is done. □

Sect. 17. Residue Theorem. In this section we assume that l is a simply connected curve. Ω is the region enclosed by l. P_1, ..., P_N are N locations in Ω. Suppose that f is analytic on $\overline{\Omega} \sim \Big\{ P_1, ..., P_N \Big\}$. Here $\sim$ is the set minus. Then we are interested in the contour integration

$$\int_l f(z)\,\mathrm{d}z, \qquad \text{where } l \text{ is counter-clockwisely oriented.}$$

Let $\epsilon > 0$ be sufficiently small radius. It is small so that $D(P_j; \epsilon)$ keeps away from l for all $j = 1, ..., N$. Moreover ϵ is small so that the closure of these N disks are mutually disjoint. By multiple connected version of Cauchy theorem, it holds

$$\int_l f(z)\,\mathrm{d}z = \sum_{j=1}^{N} \int_{\mathrm{Cir}(P_j;\epsilon)} f(z)\,\mathrm{d}z. \tag{1.110}$$

Here $\mathrm{Cir}(P_j; \epsilon)$ is also counter-clockwisely oriented. Now we compute

$$\int_{\mathrm{Cir}(P_j;\epsilon)} f(z)\,\mathrm{d}z, \qquad \text{for } j = 1, ..., N.$$

Fixing a j in $\Big\{ 1, ..., N \Big\}$, we can expand f in terms of Laurent series as follows:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - P_j)^n, \qquad \text{for } z \in \overline{D(P_j; \epsilon)} \sim \Big\{ P_j \Big\}. \tag{1.111}$$

It is clear that

$$a_{-1} = \frac{1}{2\pi i} \int_{\mathrm{Cir}(P_j;\epsilon)} f(z)\,\mathrm{d}z.$$

Equivalently

$$\int_{\mathrm{Cir}(P_j;\epsilon)} f(z)\,\mathrm{d}z = 2\pi i\, a_{-1}.$$

This equality tells us that the coefficient a_{-1} in (1.111) is crucial for us to compute the contour integration of f on $\mathrm{Cir}(P_j; \epsilon)$. In the following we give a name for a_{-1}.

Definition 1.49. *Let f be analytic on* $D(P; r) \sim \Big\{ P \Big\}$. *f satisfies*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - P)^n, \qquad \textit{for } z \in \overline{D(P; r)} \sim \Big\{ P \Big\}. \tag{1.112}$$

Then we call a_{-1} *in (1.112) the residue of f at the point P. It is denoted by* $\mathrm{Res}(f; P)$.

With this definition and the arguments above, we can rewrite (1.110) as follows:

$$\int_l f(z)\,\mathrm{d}z = 2\pi i \sum_{j=1}^{N} \mathrm{Res}(f; P_j). \tag{1.113}$$

This gives us

Theorem 1.50. *Assume that l is a simply connected curve. Ω is the region enclosed by l. P_1, ..., P_N are N locations in Ω. Suppose that f is analytic on $\overline{\Omega} \sim \left\{P_1, ..., P_N\right\}$. Then (1.113) holds.*

Theorem 1.50 shows that to compute the contour integration of f on l, we just need to do:

(i). Check if there are singularities of f in the domain enclosed by l;

(ii). If f has no singularity in the domain enclosed by l, then the contour integral of f on l is 0;

(iii). If there are singularities in the domain enclosed by l, denoted by P_1, ..., P_N, then compute $\mathrm{Res}(f; P_j)$;

(iv). The contour integration of f on l can then be evaluated by (1.113).

We now compute the residue of $f(z) = \dfrac{p(z)}{q(z)}$ at $z = z_0$. Here p and q are analytic at $z = z_0$.

Case 1. $q(z_0) \neq 0$. In this case f is also analytic at z_0. The residue of f at z_0 is 0;

Case 2. $q(z) = (z - z_0)^m \phi(z)$**, where** m **is a natural number.** ϕ **is analytic at** z_0 **with** $\phi(z_0) \neq 0$**.** In this case it holds

$$f(z) = \frac{\frac{p(z)}{\phi(z)}}{(z - z_0)^m}. \tag{1.114}$$

Letting $g(z) = \dfrac{p(z)}{\phi(z)}$, we know that g is analytic at z_0. Then g can be expanded near z_0 by the following Taylor series:

$$g(z) = \sum_{j=0}^{\infty} b_j (z - z_0)^j, \qquad \text{near } z_0. \tag{1.115}$$

Here for any $j = 0, ...$, we have

$$b_j = \frac{g^{(j)}(z_0)}{j!}. \tag{1.116}$$

Now we plug (1.115) to (1.114) and get

$$f(z) = \sum_{j=0}^{\infty} b_j (z - z_0)^{j-m}, \qquad \text{near } z_0.$$

This is the Laurent series of f near z_0. By this Laurent series, it holds

$$\mathrm{Res}(f; z_0) = b_{m-1}.$$

In light of (1.116), it follows

$$\mathrm{Res}(f; z_0) = \frac{g^{(m-1)}(z_0)}{(m-1)!}. \tag{1.117}$$

In the following we use some examples to apply the above arguments.

Example 1. In this example

$$f(z) = \frac{e^z - 1}{z^4}.$$

Now we compute $\operatorname{Res}(f;0)$. In fact we just need to let $g(z)=e^z-1$, $z_0=0$ and $m=4$ in (1.117). By this way it follows

$$\operatorname{Res}(f;0)=\frac{1}{6}.$$

Then by Residue Theorem (Theorem 1.50), it holds

$$\int_{\operatorname{Cir}(0;1)} f(z)\,\mathrm{d}z=\frac{\pi i}{3}.$$

Example 2. Evaluate

$$\int_{l_1}\frac{\mathrm{d}z}{z(z-2)^5}.$$

Here l_1 is the counter-clockwisely oriented $|z-2|=1$. Notice that in this example

$$f(z)=\frac{1}{z(z-2)^5}.$$

The region enclosed by l_1 is the closed disk $\Big\{z:|z-2|\leqslant 1\Big\}$. Clearly $z=2$ is the only singularity of f in this closed disk. Therefore by Theorem 1.50, it follows

$$\int_{l_1}\frac{\mathrm{d}z}{z(z-2)^5}=2\pi i\operatorname{Res}(f;2).$$

Now we let $g=\dfrac{1}{z}$, $z_0=2$ and $m=5$ in (1.117). Then

$$\operatorname{Res}(f;2)=\frac{1}{32}.$$

Hence the last two equalities yield

$$\int_{l_1}\frac{\mathrm{d}z}{z(z-2)^5}=\frac{\pi i}{16}.$$

Example 3. Evaluate

$$\int_{l_2}\frac{\mathrm{d}z}{z(z-2)^5}.$$

Here l_2 is the counter-clockwisely oriented $|z-2|=5$. Notice that in this example

$$f(z)=\frac{1}{z(z-2)^5}.$$

The region enclosed by l_2 is the closed disk $\Big\{z:|z-2|\leqslant 5\Big\}$. Clearly $z=0$ and $z=2$ are the only two singularities of f in this closed disk. Therefore by Theorem 1.50, it follows

$$\int_{l_2}\frac{\mathrm{d}z}{z(z-2)^5}=2\pi i\operatorname{Res}(f;2)+2\pi i\operatorname{Res}(f;0).$$

Residue of f at 2 equals to $\dfrac{1}{32}$. Now we let $g=\dfrac{1}{(z-2)^5}$, $z_0=0$ and $m=1$ in (1.117). Then

$$\operatorname{Res}(f;0)=-\frac{1}{32}.$$

Hence we get

$$\int_{l_2}\frac{\mathrm{d}z}{z(z-2)^5}=\frac{\pi i}{16}-\frac{\pi i}{16}=0.$$

Example 4. The two functions $p(z) = 1$ and $q(z) = 1 - \cos z$. Clearly the lowest order term in the Taylor expansion of $q(z)$ is z^2. Therefore we have

$$f(z) := \frac{p(z)}{q(z)} = \frac{g(z)}{z^2}, \qquad \text{where } g(z) = \frac{z^2}{1 - \cos z}.$$

Of course the residue of f at 0 is zero. On the other hand, let g in (1.117) be as in this example. Moreover we let $z_0 = 0$ and $m = 2$ there. So the residue of f at 0 can also be calculated as follows:

$$\lim_{z \to 0} g'(z) = \lim_{z \to 0} \frac{2z\big(1 - \cos z\big) - z^2 \sin z}{(1 - \cos z)^2} = 0.$$

Example 5. Consider the function

$$f(z) = \cot z = \frac{\cos z}{\sin z}.$$

At $n\pi$ where n is an integer, $\sin z = 0$. Therefore we can have

$$f(z) = \frac{g(z)}{z - n\pi}, \qquad \text{where } g(z) = \cos z \, \frac{z - n\pi}{\sin z}.$$

Now we let g in (1.117) as in this example and let $z_0 = n\pi$, $m = 1$ there. Therefore it holds

$$\text{Res}(f; n\pi) = \lim_{z \to n\pi} g(z) = 1.$$

Example 6. Consider the function

$$f(z) = \frac{z - \sinh z}{z^2 \sin hz}.$$

Notice that $z^2 \sinh z = 0$ implies $z = 0$ or $z = n\pi i$, where n is an integer. Now we compute the residue of f at these locations. Firstly we consider $n\pi i$ where $n \neq 0$. As before we can rewrite f as

$$f(z) = \frac{g(z)}{z - n\pi i}, \qquad \text{where } g(z) = \frac{z - \sinh z}{z^2} \frac{z - n\pi i}{\sinh z}.$$

Then by (1.117), it holds

$$\text{Res}(f; n\pi i) = \lim_{z \to n\pi i} g(z) = \frac{1}{n\pi i} \frac{1}{\cosh n\pi i} = \frac{(-1)^n}{n\pi i}.$$

Now we consider the residue at 0. In fact $\sinh z$ near 0 can be written as follows:

$$\sinh z = z \, \frac{\sinh z}{z}.$$

The definition of $\frac{\sinh z}{z}$ can be extended to the location 0. If we use h to denote the extension of $\frac{\sinh z}{z}$ on $\mathbb{C}$, then clearly $h(0) = 1 \neq 0$. Plugging $\sinh z = zh(z)$ into the definition of f yields

$$f(z) = \frac{1 - h(z)}{z^2 h(z)}.$$

By L'Hospital's rule, it holds

$$\lim_{z \to 0} \frac{1 - h(z)}{z^2} = \lim_{z \to 0} \frac{z - \sinh z}{z^3} = \lim_{z \to 0} \frac{1 - \cosh z}{3z^2} = -\frac{1}{6} \lim_{z \to 0} \frac{\sinh z}{z} = -\frac{1}{6} \lim_{z \to 0} \cosh z = -\frac{1}{6}.$$

Therefore f can be analytically extended to the origin. Indeed we can redefine the value of f at 0 to be $-\frac{1}{6}$ so that the extended f is analytic at 0. By this way the residue of f at 0 equals to 0.

Sect. 18. Improper Integrals. Improper integrals are integrals for real-valued functions on $\mathbb{R}$ or $\mathbb{R}^+ = \Big\{x : x \geqslant 0\Big\}$. In terms of proper integrals, we define

$$\int_0^\infty f(x)\,\mathrm{d}x = \lim_{R\to\infty}\int_0^R f(x)\,\mathrm{d}x.$$

As for the improper integrals on $\mathbb{R}$, we use the following principal way to define the corresponding improper integrals:

$$\text{P.V.}\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \lim_{R\to\infty}\int_{-R}^R f(x)\,\mathrm{d}x.$$

In this section we use Residue theorem to evaluate four types of improper integrals.

Sect. 18.1, Type I. Type I integrals are for rational functions.

Example 1. Evaluate the integral:

$$\int_0^\infty \frac{\mathrm{d}x}{x^6+1}.$$

Firstly we pick up a contour. Let C_R be the upper-half circle with center 0 and radius R. Moreover C_R is counter-clockwisely oriented. Then we get a contour l_R by moving from $-R$ to R along the real axis and then moving from R back to $-R$ along C_R. The region enclosed by l_R is denoted by D_R^+. Clearly it is the upper-half part of the closed disk $\Big\{z : |z| \leqslant R\Big\}$. In D_R^+, there are three zeros of z^6+1. They are

$$c_0 = e^{i\pi/6}, \qquad c_1 = i, \qquad c_2 = e^{i5\pi/6}.$$

Therefore by Residue theorem, it follows

$$\int_{l_R}\frac{\mathrm{d}z}{z^6+1} = 2\pi i\left(\mathrm{Res}\left(\frac{1}{z^6+1}; c_0\right) + \mathrm{Res}\left(\frac{1}{z^6+1}; c_1\right) + \mathrm{Res}\left(\frac{1}{z^6+1}; c_2\right)\right). \tag{1.118}$$

For any $k = 0, 1, 2$, it holds

$$\mathrm{Res}\left(\frac{1}{z^6+1}; c_k\right) = \lim_{z\to c_k}\frac{z-c_k}{z^6+1} = \lim_{z\to c_k}\frac{1}{6z^5} = \frac{1}{6c_k^5} = \frac{c_k}{6c_k^6} = -\frac{c_k}{6}.$$

Applying this result to the right-hand side of (1.118) yields

$$\int_{l_R}\frac{\mathrm{d}z}{z^6+1} = \int_{-R}^R\frac{\mathrm{d}x}{x^6+1} + \int_{C_R}\frac{\mathrm{d}z}{z^6+1} = \frac{2\pi}{3}.$$

Equivalently it gives us

$$\int_{-R}^R\frac{\mathrm{d}x}{x^6+1} = \frac{2\pi}{3} - \int_{C_R}\frac{\mathrm{d}z}{z^6+1}. \tag{1.119}$$

As for the last integral above, it holds

$$\left|\int_{C_R}\frac{\mathrm{d}z}{z^6+1}\right| \leqslant \int_{C_R}\frac{|\mathrm{d}z|}{R^6-1} = \frac{2\pi R}{R^6-1} \longrightarrow 0, \qquad \text{as } R\to\infty.$$

By this limit and taking $R\to\infty$ in (1.119), we have

$$\text{P.V.}\int_{-\infty}^\infty\frac{\mathrm{d}x}{x^6+1} = \frac{2\pi}{3}.$$

Since the integrand is even, we know from the above result that

$$\int_0^\infty \frac{\mathrm{d}x}{x^6+1} = \frac{\pi}{3}.$$

Sect. 18.2. Type II. Improper Integrals from Fourier Analysis. In this section we consider

$$\int_{-\infty}^{\infty} f(x)\sin ax\,\mathrm{d}x \quad\text{and}\quad \int_{-\infty}^{\infty} f(x)\cos ax\,\mathrm{d}x, \tag{1.120}$$

where a is a positive constant. In terms of the Euler's formula, we can equivalently consider

$$\int_{-\infty}^{\infty} f(x)e^{iax}\,\mathrm{d}x. \tag{1.121}$$

Now we pick up the same contour l_R as in the previous section. By Residue theorem it follows

$$\int_{l_R} f(z)e^{iaz}\,\mathrm{d}z = \int_{-R}^{R} f(z)e^{iaz}\,\mathrm{d}z + \int_{C_R} f(z)e^{iaz}\,\mathrm{d}z = 2\pi i\sum_{j=1}^{N}\mathrm{Res}\left(f(z)e^{iaz};P_j\right).$$

Here we denote by P_j $(j = 1, ..., N)$ the N singularities of $f(z)e^{iaz}$ in D_R, for large R. By the last equality, we get

$$\int_{-R}^{R} f(z)e^{iaz}\,\mathrm{d}z = 2\pi i\sum_{j=1}^{N}\mathrm{Res}\left(f(z)e^{iaz};P_j\right) - \int_{C_R} f(z)e^{iaz}\,\mathrm{d}z. \tag{1.122}$$

Therefore to evaluate (1.121), we need residue of $f(z)e^{iaz}$ at each P_j. Moreover we also need to check the limit

$$\lim_{R\to\infty}\int_{C_R} f(z)e^{iaz}\,\mathrm{d}z. \tag{1.123}$$

Example 2. Evaluate

$$\int_0^\infty \frac{\cos 2x}{\left(x^2+4\right)^2}\,\mathrm{d}x. \tag{1.124}$$

Letting $f(z) = \dfrac{1}{\left(z^2+4\right)^2}$ and $a = 2$ in (1.122), we get

$$\int_{-R}^{R} \frac{e^{i2x}}{\left(x^2+4\right)^2}\,\mathrm{d}x = 2\pi i\sum_{j=1}^{N}\mathrm{Res}\left(\frac{e^{i2z}}{\left(z^2+4\right)^2};P_j\right) - \int_{C_R}\frac{e^{i2z}}{\left(z^2+4\right)^2}\,\mathrm{d}z.$$

Clearly $2i$ is the only singularity of $\dfrac{e^{i2z}}{\left(z^2+4\right)^2}$ in D_R for R large. Therefore the last equality is reduced to

$$\int_{-R}^{R} \frac{e^{i2x}}{\left(x^2+4\right)^2}\,\mathrm{d}x = 2\pi i\mathrm{Res}\left(\frac{e^{i2z}}{\left(z^2+4\right)^2};2i\right) - \int_{C_R}\frac{e^{i2z}}{\left(z^2+4\right)^2}\,\mathrm{d}z, \qquad \text{for large } R. \tag{1.125}$$

On one hand we have

$$\frac{e^{i2z}}{\left(z^2+4\right)^2} = \frac{g(z)}{(z-2i)^2}, \qquad \text{where } g(z) = \frac{e^{i2z}}{(z+2i)^2}.$$

By (1.117), it follows

$$\mathrm{Res}\left(\frac{e^{i2z}}{\left(z^2+4\right)^2};2i\right) = g'(2i) = \frac{5}{32i}\,e^{-4}. \tag{1.126}$$

On the other hand we have

$$\left|\int_{C_R} \frac{e^{i2z}}{\left(z^2+4\right)^2}\,\mathrm{d}z\right| \leqslant \int_{C_R} \left|\frac{e^{i2(x+iy)}}{\left(z^2+4\right)^2}\right| |\,\mathrm{d}z\,| \leqslant \int_{C_R} \frac{e^{-2y}}{\left(R^2-4\right)^2}\,|\,\mathrm{d}z\,|.$$

Notice that on C_R, the y-variable is non-negative. Therefore the last estimate can be reduced to

$$\left|\int_{C_R} \frac{e^{i2z}}{\left(z^2+4\right)^2}\,\mathrm{d}z\right| \leqslant \int_{C_R} \frac{1}{\left(R^2-4\right)^2}\,|\,\mathrm{d}z\,| = \frac{2\pi R}{\left(R^2-4\right)^2} \longrightarrow 0, \qquad \text{as } R\to\infty.$$

Applying this limit together with (1.126) to the right-hand side of (1.125), we have

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{e^{i2x}}{\left(x^2+4\right)^2}\,\mathrm{d}x = \lim_{R\to\infty}\int_{-R}^{R} \frac{e^{i2x}}{\left(x^2+4\right)^2}\,\mathrm{d}x = \frac{5\pi}{16e^4}.$$

Taking real part on both sides above yields

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{\cos 2x}{\left(x^2+4\right)^2}\,\mathrm{d}x = \frac{5\pi}{16e^4}.$$

By even symmetry of the integrand above, it follows

$$\int_{0}^{\infty} \frac{\cos 2x}{\left(x^2+4\right)^2}\,\mathrm{d}x = \frac{5\pi}{32e^4}.$$

One can see that it is important to show that the limit in (1.123) equals to 0. Now we give a general result:

Lemma 1.51 (Jordan's lemma)**.** *Suppose that*

(a). a function $f(z)$ is analytic at all points in the upper-half plane $y \geqslant 0$ that are exterior to a circle $|\,z\,| = R_0$;

(b). C_R denotes a semicircle $z = Re^{i\theta}$ $(0 \leqslant \theta \leqslant \pi)$, where $R > R_0$;

(c). for all points z on C_R, there is a positive constant M_R so that

$$|\,f(z)\,| \leqslant M_R \quad \textit{and} \quad \lim_{R\to\infty} M_R = 0.$$

Then the limit in (1.123) equals to 0.

Proof. Since

$$\int_{C_R} f(z)e^{iaz}\,\mathrm{d}z = \int_0^{\pi} f\left(Re^{i\theta}\right) e^{iaRe^{i\theta}} Re^{i\theta} i\,\mathrm{d}\theta = iR\int_0^{\pi} f\left(Re^{i\theta}\right) e^{-aR\sin\theta} e^{iaR\cos\theta} e^{i\theta}\,\mathrm{d}\theta,$$

by (c) in the hypothesis of the lemma, it follows

$$\left|\int_{C_R} f(z)e^{iaz}\,\mathrm{d}z\right| \leqslant RM_R\int_0^{\pi} e^{-aR\sin\theta}\,\mathrm{d}\theta. \tag{1.127}$$

Now we consider

$$\int_0^{\pi} e^{-aR\sin\theta}\,\mathrm{d}\theta.$$

Clearly it can be separated into

$$\int_0^{\pi} e^{-aR\sin\theta}\,\mathrm{d}\theta = \int_0^{\pi/4} e^{-aR\sin\theta}\,\mathrm{d}\theta + \int_{\pi/4}^{3\pi/4} e^{-aR\sin\theta}\,\mathrm{d}\theta + \int_{3\pi/4}^{\pi} e^{-aR\sin\theta}\,\mathrm{d}\theta. \tag{1.128}$$

For the first integral on the right-hand side of (1.128), since

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1,$$

there is a positive constant c_0 so that

$$\frac{\sin\theta}{\theta} \geqslant c_0, \qquad \text{on } [0, \pi/4].$$

By this inequality, it holds

$$\int_0^{\pi/4} e^{-aR\sin\theta}\,\mathrm{d}\theta \leqslant \int_0^{\pi/4} e^{-ac_0R\theta}\,\mathrm{d}\theta = -\frac{e^{-ac_0R\theta}}{ac_0R}\Big|_0^{\pi/4} \leqslant \frac{1}{ac_0R}. \tag{1.129}$$

As for the second integral on the right-hand side of (1.128), it holds

$$\int_{\pi/4}^{3\pi/4} e^{-aR\sin\theta}\,\mathrm{d}\theta \leqslant \int_{\pi/4}^{3\pi/4} e^{-\sqrt{2}aR/2}\,\mathrm{d}\theta = \frac{\pi e^{-\sqrt{2}aR/2}}{2}. \tag{1.130}$$

Here we used $\sin\theta \geqslant \sqrt{2}/2$ on $[\pi/4, 3\pi/4]$. For the third integral on the right-hand side of (1.128), we apply change of variable $\alpha = \pi - \theta$ and get

$$\int_{3\pi/4}^{\pi} e^{-aR\sin\theta}\,\mathrm{d}\theta = \int_0^{\pi/4} e^{-aR\sin(\pi-\alpha)}\,\mathrm{d}\alpha = \int_0^{\pi/4} e^{-aR\sin\alpha}\,\mathrm{d}\alpha.$$

(1.129) can then be applied. Summarizing the above arguments, we have

$$\int_0^{\pi} e^{-aR\sin\theta}\,\mathrm{d}\theta \leqslant \frac{2}{ac_0R} + \frac{\pi e^{-\sqrt{2}aR/2}}{2}.$$

Plugging this estimate into the right-hand side of (1.127) yields

$$\left|\int_{C_R} f(z)e^{iaz}\,\mathrm{d}z\right| \leqslant \frac{2M_R}{ac_0} + RM_R\frac{\pi e^{-\sqrt{2}aR/2}}{2} \longrightarrow 0, \qquad \text{as } R \to \infty.$$

The convergence above holds by (c) in the hypothesis of this lemma. The proof is finished. □

Example 3. Evaluate

$$\int_0^{\infty} \frac{x\sin 2x}{x^2+3}\,\mathrm{d}x. \tag{1.131}$$

Letting $f(z) = \dfrac{z}{z^2+3}$ and $a = 2$ in (1.122), we get

$$\int_{-R}^{R} \frac{xe^{i2x}}{x^2+3}\,\mathrm{d}x = 2\pi i\sum_{j=1}^{N}\mathrm{Res}\left(f(z)e^{i2z}; P_j\right) - \int_{C_R}\frac{ze^{i2z}}{z^2+3}\,\mathrm{d}z.$$

Clearly $\sqrt{3}i$ is the only singularity of $f(z)e^{i2z}$ in D_R for R large. Therefore the last equality is reduced to

$$\int_{-R}^{R} \frac{xe^{i2x}}{x^2+3}\,\mathrm{d}x = 2\pi i\mathrm{Res}\Big(f(z)e^{i2z}; \sqrt{3}i\Big) - \int_{C_R}\frac{ze^{i2z}}{z^2+3}\,\mathrm{d}z, \qquad \text{for large } R. \tag{1.132}$$

On one hand we have

$$f(z)e^{i2z} = \frac{g(z)}{z-\sqrt{3}i}, \qquad \text{where } g(z) = \frac{ze^{i2z}}{z+\sqrt{3}i}.$$

By (1.117), it follows

$$\mathrm{Res}\Big(f(z)e^{i2z}; \sqrt{3}i\Big) = g(\sqrt{3}i) = \frac{1}{2e^{2\sqrt{3}}}. \tag{1.133}$$

On the other hand we have

$$|\,f(z)\,| \leqslant \frac{R}{R^2-3}, \qquad \text{on } C_R.$$

Therefore (c) in Lemma 1.51 is fulfilled. By Jordan's lemma and (1.133), we can take $R \to \infty$ in (1.132) and get

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{xe^{i2x}}{x^2+3}\, \mathrm{d}x = \lim_{R\to\infty} \int_{-R}^{R} \frac{xe^{i2x}}{x^2+3}\, \mathrm{d}x = \pi i e^{-2\sqrt{3}}.$$

Taking imaginary part above yields

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x\sin 2x}{x^2+3}\, \mathrm{d}x = \pi e^{-2\sqrt{3}}.$$

By the even symmetry of the integrand above, it follows

$$\int_{0}^{\infty} \frac{x\sin 2x}{x^2+3}\, \mathrm{d}x = \frac{\pi}{2} e^{-2\sqrt{3}}.$$

Sect. 18.3. Type III. Integrals involving indented path. Let $0 < \rho < R$. C_R is the same as the previous sections. Moreover we let c_ρ be the upper-half circle $|z| = \rho$. Different from C_R, we let C_ρ clockwisely oriented. Now we denote by $l_{\rho,R}$ the path starting from $-R$ to $-\rho$ along the real axis, then from $-\rho$ to ρ along C_ρ, then from ρ to R along the real axis and finally from R back to $-R$ along C_R. We refer $l_{\rho,R}$ as a indented path. To construct such contour is to avoid difficulty from singularity at 0. For example e^{ix}/x. This function does not in general have definition at 0.

Example 4. Evaluate the Dirichlet's integral

$$\int_0^{\infty} \frac{\sin x}{x}\, \mathrm{d}x.$$

By Residue theorem and the indented path $l_{\rho,R}$, we have

$$\int_{l_{\rho,R}} \frac{e^{iz}}{z}\, \mathrm{d}z = \int_{-R}^{-\rho} \frac{e^{ix}}{x}\, \mathrm{d}z + \int_{C_\rho} \frac{e^{iz}}{z}\, \mathrm{d}z + \int_{\rho}^{R} \frac{e^{ix}}{x}\, \mathrm{d}x + \int_{C_R} \frac{e^{iz}}{z}\, \mathrm{d}z = 0. \tag{1.134}$$

Here the function e^{iz}/z has no singularity in the region enclosed by $l_{\rho,R}$. Rewriting (1.134) yields

$$\int_{-R}^{-\rho} \frac{e^{ix}}{x}\, \mathrm{d}z + \int_{\rho}^{R} \frac{e^{ix}}{x}\, \mathrm{d}x = -\int_{C_\rho} \frac{e^{iz}}{z}\, \mathrm{d}z - \int_{C_R} \frac{e^{iz}}{z}\, \mathrm{d}z. \tag{1.135}$$

Notice that C_ρ is clockwisely oriented. Hence we parameterize it by $\rho e^{i(\pi-\theta)}$ with $\theta \in [0,\pi]$. Therefore

$$\int_{C_\rho} \frac{e^{iz}}{z}\, \mathrm{d}z = -\int_0^{\pi} \frac{e^{i\rho e^{i(\pi-\theta)}}}{\rho e^{i(\pi-\theta)}} \rho e^{i(\pi-\theta)} i\, \mathrm{d}\theta = -i \int_0^{\pi} e^{-i\rho\cos\theta} e^{-\rho\sin\theta}\, \mathrm{d}\theta. \tag{1.136}$$

Since for any $z \in D(0;1)$, it holds

$$\left| e^z - 1 \right| \leqslant c\,|z|, \qquad \text{for some constant } c > 0.$$

Therefore for all $\rho \in (0,1)$, it holds

$$\left| e^{-i\rho\cos\theta - \rho\sin\theta} - 1 \right| \leqslant c\,\rho.$$

By this estimate, it follows

$$\left| \int_0^{\pi} e^{-i\rho\cos\theta} e^{-\rho\sin\theta} - 1\, \mathrm{d}\theta \right| \leqslant \int_0^{\pi} \left| e^{-i\rho\cos\theta} e^{-\rho\sin\theta} - 1 \right| \mathrm{d}\theta \leqslant c\,\pi\,\rho, \qquad \text{for all } \rho \in (0,1).$$

By the last estimate, we can take $\rho \to 0^+$ and get

$$\lim_{\rho\to 0^+} \int_0^{\pi} e^{-i\rho\cos\theta} e^{-\rho\sin\theta}\, \mathrm{d}\theta = \pi.$$

With this limit, we can take $\rho \to 0^+$ in (1.136) and get

$$\lim_{\rho\to 0^+} \int_{C_\rho} \frac{e^{iz}}{z}\, \mathrm{d}z = -\pi i.$$

With this limit, we have from (1.135) that

$$\int_{-R}^{0} \frac{e^{ix}}{x}\, \mathrm{d}z + \int_0^R \frac{e^{ix}}{x}\, \mathrm{d}x = \pi i - \int_{C_R} \frac{e^{iz}}{z}\, \mathrm{d}z. \tag{1.137}$$

Here we take $\rho \to 0$ in (1.135). Moreover by Jordan's lemma, one can easily show that

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{z}\, \mathrm{d}z = 0.$$

By this limit, we can take $R \to \infty$ in (1.137) and get

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, \mathrm{d}z = \pi i.$$

Taking imaginary part above and noticing the even symmetry of $\sin x/x$ on $\mathbb{R}$, we have

$$\int_0^{\infty} \frac{\sin x}{x}\, \mathrm{d}x = \frac{\pi}{2}.$$

Example 5. Evaluate for $-1 < a < 3$ the integration:

$$\int_0^{\infty} \frac{x^a}{(x^2+1)^2}\, \mathrm{d}x.$$

We let

$$f(z) = \frac{z^a}{(z^2+1)^2}, \qquad \text{where } z^a \text{ is the power function defined in the branch } -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}.$$

With the contour $l_{\rho,R}$ and residue theorem, it holds

$$\int_{l_{\rho,R}} f(z)\, \mathrm{d}z = \int_{-R}^{-\rho} \frac{z^a}{(z^2+1)^2}\, \mathrm{d}z + \int_{C_\rho} \frac{z^a}{(z^2+1)^2}\, \mathrm{d}z + \int_{\rho}^{R} \frac{z^a}{(z^2+1)^2}\, \mathrm{d}z + \int_{C_R} \frac{z^a}{(z^2+1)^2}\, \mathrm{d}z = 2\pi i \operatorname{Res}(f(z); i). \tag{1.138}$$

Here we have used the fact that i is the only singularity of $f(z)$ in the region enclosed by $l_{\rho,R}$, provided that ρ is small and R is large.

(i). The function f can be rewritten as

$$f(z) = \frac{g(z)}{(z-i)^2}, \qquad \text{where } g(z) = \frac{z^a}{(z+i)^2}.$$

Hence by (1.117), it holds

$$\operatorname{Res}(f(z); i) = g'(i) = \frac{(a-1)i^{a+1}}{4} = \frac{a-1}{4} e^{\pi(a+1)i/2}. \tag{1.139}$$

(ii). Letting $z=-t$ with t running from R to ρ, we then have

$$\int_{-R}^{-\rho} \frac{z^a}{(z^2+1)^2}\,\mathrm{d}z = -\int_{R}^{\rho} \frac{(-t)^a}{(t^2+1)^2}\,\mathrm{d}t = \int_{\rho}^{R} \frac{e^{a\log(-t)}}{(t^2+1)^2}\,\mathrm{d}t.$$

Using the branch of log-function, we calculate $\log(-t) = \ln t + i\arg(-t) = \ln t + i\pi$. Plugging this calculation into the last equality yields

$$\int_{-R}^{-\rho} \frac{z^a}{(z^2+1)^2}\,\mathrm{d}z = \int_{\rho}^{R} \frac{e^{a\log(-t)}}{(t^2+1)^2}\,\mathrm{d}t = e^{ia\pi}\int_{\rho}^{R} \frac{t^a}{(t^2+1)^2}\,\mathrm{d}t. \tag{1.140}$$

(iii). Similarly we let $z=t$ with t running from ρ to R. Then it holds

$$\int_{\rho}^{R} \frac{z^a}{(z^2+1)^2}\,\mathrm{d}z = \int_{\rho}^{R} \frac{t^a}{(t^2+1)^2}\,\mathrm{d}t \tag{1.141}$$

(iv). Now we let $z(\theta)=\rho e^{i(\pi-\theta)}$, where θ runs from 0 to π. By this parametrization, it follows

$$\int_{C_\rho} \frac{z^a}{(z^2+1)^2}\,\mathrm{d}z = -i\int_0^{\pi} \frac{e^{a\log\left(\rho e^{i(\pi-\theta)}\right)}}{(\rho^2 e^{2i(\pi-\theta)}+1)^2}\rho e^{i(\pi-\theta)}\,\mathrm{d}\theta = -i\,\rho\int_0^{\pi} \frac{e^{a(\ln\rho+i(\pi-\theta))}}{(\rho^2 e^{2i(\pi-\theta)}+1)^2}\,e^{i(\pi-\theta)}\,\mathrm{d}\theta.$$

Reorganizing the above calculations yields

$$\int_{C_\rho} \frac{z^a}{(z^2+1)^2}\,\mathrm{d}z = -i\,\rho^{1+a}\int_0^{\pi} \frac{e^{(a+1)i(\pi-\theta)}}{(\rho^2 e^{2i(\pi-\theta)}+1)^2}\,\mathrm{d}\theta.$$

Taking ρ sufficiently small and applying triangle inequality, we get

$$\left|\int_{C_\rho} \frac{z^a}{(z^2+1)^2}\,\mathrm{d}z\right| \leqslant \rho^{1+a}\int_0^{\pi} \frac{1}{(1-\rho^2)^2}\,\mathrm{d}\theta = \pi\frac{\rho^{1+a}}{(1-\rho^2)^2}.$$

Since $a+1>0$, the last estimate gives us

$$\lim_{\rho\to 0}\int_{C_\rho} \frac{z^a}{(z^2+1)^2}\,\mathrm{d}z = 0. \tag{1.142}$$

(v). let $z(\theta)=Re^{i\theta}$, where θ runs from 0 to π. By this parametrization, it follows

$$\int_{C_R} \frac{z^a}{(z^2+1)^2}\,\mathrm{d}z = i\int_0^{\pi} \frac{e^{a\log\left(Re^{i\theta}\right)}}{(R^2e^{2i\theta}+1)^2}Re^{i\theta}\,\mathrm{d}\theta = i\,R\int_0^{\pi} \frac{e^{a(\ln R+i\theta)}}{(R^2e^{2i\theta}+1)^2}\,e^{i\theta}\,\mathrm{d}\theta.$$

Reorganizing the above calculations yields

$$\int_{C_R} \frac{z^a}{(z^2+1)^2}\,\mathrm{d}z = i\,R^{1+a}\int_0^{\pi} \frac{e^{(a+1)i\theta}}{(R^2e^{2i\theta}+1)^2}\,\mathrm{d}\theta.$$

Taking R sufficiently large and applying triangle inequality, we get

$$\left|\int_{C_R} \frac{z^a}{(z^2+1)^2}\,\mathrm{d}z\right| \leqslant R^{1+a}\int_0^{\pi} \frac{1}{(R^2-1)^2}\,\mathrm{d}\theta = \pi\frac{R^{1+a}}{(R^2-1)^2}.$$

Since $a+1<4$, the last estimate gives us

$$\lim_{R\to\infty}\int_{C_R} \frac{z^a}{(z^2+1)^2}\,\mathrm{d}z = 0. \tag{1.143}$$

Now we plug (1.139)-(1.141) to (1.138) and obtain

$$\left(e^{ia\pi}+1\right)\int_{\rho}^{R} \frac{t^a}{(t^2+1)^2}\,\mathrm{d}t = 2\pi i\,\frac{a-1}{4}e^{\pi(a+1)i/2} - \int_{C_\rho} \frac{z^a}{(z^2+1)^2}\,\mathrm{d}z - \int_{C_R} \frac{z^a}{(z^2+1)^2}\,\mathrm{d}z.$$

In light of (1.142)-(1.143), we can take $\rho \to 0$ and $R \to \infty$ in the last equality and get

$$\left(e^{ia\pi}+1\right)\int_0^\infty \frac{t^a}{(t^2+1)^2}\,\mathrm{d}t = 2\pi i\,\frac{a-1}{4}e^{\pi(a+1)i/2},$$

which implies

$$\int_0^\infty \frac{t^a}{(t^2+1)^2}\,\mathrm{d}t = 2\pi i\,\frac{a-1}{4}\frac{e^{\pi(a+1)i/2}}{e^{ia\pi}+1} = \frac{\pi(1-a)}{4\cos\left(\frac{a\pi}{2}\right)}, \qquad \text{when } a \neq 1.$$

For $a = 1$, this integral can be calculated directly by change of variable. In fact

$$\begin{aligned}\int_0^\infty \frac{t}{(t^2+1)^2}\,\mathrm{d}t &= \lim_{R\to\infty}\int_0^R \frac{t}{(t^2+1)^2}\,\mathrm{d}t = \lim_{R\to\infty}\frac{1}{2}\int_0^R \frac{\mathrm{d}\left(t^2+1\right)}{(t^2+1)^2}\\ &= \lim_{R\to\infty} -\frac{1}{2(t^2+1)}\Bigg|_0^R = \lim_{R\to\infty} -\frac{1}{2(t^2+1)}\Bigg|_0^R = \lim_{R\to\infty}\frac{1}{2}\frac{R^2}{1+R^2} = \frac{1}{2}.\end{aligned}$$

Sect. 18.4 Integration Along a Branch Cut. The example in this section is to calculate the integral

$$\int_0^\infty \frac{x^{-a}}{x+1}\,\mathrm{d}x, \qquad \text{where } 0 < a < 1. \tag{1.144}$$

When we complexify the integrand to the complex function

$$\frac{z^{-a}}{z+1}, \tag{1.145}$$

we need to fix a branch cut. In this example we let $0 < \arg z < 2\pi$. Therefore the branch cut is the positive part of the real axis. Clearly the function in (1.145) is not analytic on any point of the branch cut. Therefore when we construct contour, we have to avoid touching the branch cut. Now we fix a $\rho > 0$ sufficiently small and fix a $R > 0$ sufficiently large. Fixing a $\theta_0 > 0$ sufficiently small, we have two rays. One is denoted by l_+ which has argument θ_0. Another ray is denoted by l_- which has argument $-\theta_0$. l_+ intersects with $\mathrm{Cir}(0;\rho)$ at A and intersects with $\mathrm{Cir}(0;R)$ at B. Similarly l_- intersects with $\mathrm{Cir}(0;\rho)$ at C and intersects with $\mathrm{Cir}(0;R)$ at D. Now we construct our contour l. Starting from A, we follow l_+ to B. This part of contour is denoted by l_1. Then we go from B to D counter-clockwisely along the circle $\mathrm{Cir}(0;R)$. This part of contour is denoted by l_2. From D to C, we follow the l_-. This part of contour is denoted by l_3. Finally we go from C back to A by clockwisely along $\mathrm{Cir}(0;\rho)$. This part of contour is denoted by l_4. Therefore by residue theorem, it follows

$$\int_l \frac{z^{-a}}{z+1}\,\mathrm{d}z = \int_{l_1}\frac{z^{-a}}{z+1}\,\mathrm{d}z + \int_{l_2}\frac{z^{-a}}{z+1}\,\mathrm{d}z + \int_{l_3}\frac{z^{-a}}{z+1}\,\mathrm{d}z + \int_{l_4}\frac{z^{-a}}{z+1}\,\mathrm{d}z = 2\pi i\,\mathrm{Res}\left(\frac{z^{-a}}{z+1};-1\right). \tag{1.146}$$

(i). By (1.117), it holds

$$\mathrm{Res}\left(\frac{z^{-a}}{z+1};-1\right) = (-1)^{-a} = e^{-ia\pi}. \tag{1.147}$$

(ii). The parametrization for l_1 is $re^{i\theta_0}$ where r runs from ρ to R. Hence

$$\int_{l_1}\frac{z^{-a}}{z+1}\,\mathrm{d}z = \int_\rho^R \frac{e^{-a\log(re^{i\theta_0})}}{re^{i\theta_0}+1}\,e^{i\theta_0}\mathrm{d}r = \int_\rho^R \frac{e^{-a(\ln r + i\theta_0)}}{re^{i\theta_0}+1}\,e^{i\theta_0}\mathrm{d}r = e^{(1-a)i\theta_0}\int_\rho^R \frac{r^{-a}}{re^{i\theta_0}+1}\,\mathrm{d}r. \tag{1.148}$$

(iii). The parametrization for l_2 is $Re^{i\theta}$ where θ runs from θ_0 to $2\pi - \theta_0$. Hence

$$\int_{l_2}\frac{z^{-a}}{z+1}\,\mathrm{d}z = iR\int_{\theta_0}^{2\pi-\theta_0}\frac{e^{-a\log(Re^{i\theta})}}{Re^{i\theta}+1}\,e^{i\theta}\,\mathrm{d}\theta = iR\int_{\theta_0}^{2\pi-\theta_0}\frac{e^{-a(\ln R + i\theta)}}{Re^{i\theta}+1}\,e^{i\theta}\,\mathrm{d}\theta.$$

It then holds

$$\int_{l_2} \frac{z^{-a}}{z+1}\,\mathrm{d}z = iR^{1-a}\int_{\theta_0}^{2\pi-\theta_0} \frac{e^{(1-a)i\theta}}{Re^{i\theta}+1}\,\mathrm{d}\theta.$$

Notice that $a > 0$. Therefore

$$\left|\int_{l_2} \frac{z^{-a}}{z+1}\,\mathrm{d}z\right| \leqslant R^{1-a}\int_{\theta_0}^{2\pi-\theta_0} \frac{1}{R-1}\,\mathrm{d}\theta \leqslant 2\pi\frac{R^{1-a}}{R-1} \longrightarrow 0, \qquad \text{as } R \to \infty. \tag{1.149}$$

(iv). The parametrization for l_3 is $re^{i(2\pi-\theta_0)}$ where r runs from R to ρ. Hence

$$\begin{aligned}\int_{l_3} \frac{z^{-a}}{z+1}\,\mathrm{d}z &= \int_R^\rho \frac{e^{-a\log(re^{i(2\pi-\theta_0)})}}{re^{i(2\pi-\theta_0)}+1}\,e^{i(2\pi-\theta_0)}\mathrm{d}r \\ &= \int_R^\rho \frac{e^{-a(\ln r+i(2\pi-\theta_0))}}{re^{i(2\pi-\theta_0)}+1}\,e^{i(2\pi-\theta_0)}\mathrm{d}r = e^{(1-a)i(2\pi-\theta_0)}\int_R^\rho \frac{r^{-a}}{re^{i(2\pi-\theta_0)}+1}\,\mathrm{d}r.\end{aligned} \tag{1.150}$$

(v). The parametrization for l_4 is $\rho e^{i\theta}$ where θ runs from $2\pi-\theta_0$ to θ_0. Hence

$$\int_{l_4} \frac{z^{-a}}{z+1}\,\mathrm{d}z = i\rho\int_{2\pi-\theta_0}^{\theta_0} \frac{e^{-a\log(\rho e^{i\theta})}}{\rho e^{i\theta}+1}\,e^{i\theta}\,\mathrm{d}\theta = i\rho\int_{2\pi-\theta_0}^{\theta_0} \frac{e^{-a(\ln\rho+i\theta)}}{\rho e^{i\theta}+1}\,e^{i\theta}\,\mathrm{d}\theta.$$

It then holds

$$\int_{l_4} \frac{z^{-a}}{z+1}\,\mathrm{d}z = i\rho^{1-a}\int_{2\pi-\theta_0}^{\theta_0} \frac{e^{(1-a)i\theta}}{\rho e^{i\theta}+1}\,\mathrm{d}\theta.$$

Notice that $1-a > 0$. Therefore

$$\left|\int_{l_4} \frac{z^{-a}}{z+1}\,\mathrm{d}z\right| \leqslant \rho^{1-a}\int_{\theta_0}^{2\pi-\theta_0} \frac{1}{1-\rho}\,\mathrm{d}\theta \leqslant 2\pi\frac{\rho^{1-a}}{1-\rho} \longrightarrow 0, \qquad \text{as } \rho \to 0. \tag{1.151}$$

Applying (1.147)-(1.151) to (1.146) and then taking $\rho \to 0$, $R \to \infty$, we have

$$e^{(1-a)i\theta_0}\int_0^\infty \frac{r^{-a}}{re^{i\theta_0}+1}\,\mathrm{d}r + e^{(1-a)i(2\pi-\theta_0)}\int_\infty^0 \frac{r^{-a}}{re^{i(2\pi-\theta_0)}+1}\,\mathrm{d}r = 2\pi i\,e^{-ia\pi}.$$

Finally we take $\theta_0 \to 0^+$ above and get

$$\left(1-e^{2\pi(1-a)i}\right)\int_0^\infty \frac{r^{-a}}{r+1}\,\mathrm{d}r = 2\pi i\,e^{-ia\pi},$$

which shows that

$$\int_0^\infty \frac{r^{-a}}{r+1}\,\mathrm{d}r = \frac{\pi}{\sin a\pi}.$$

Sect. 19. Definite Integral. The residue theorem can also help us to calculate definite integrals involving sin and cos functions. In fact we consider the integral of the type

$$\int_0^{2\pi} F(\sin\theta,\cos\theta)\,\mathrm{d}\theta. \tag{1.152}$$

If we let $z(\theta) = e^{i\theta}$ with $\theta \in [0, 2\pi]$, then we have $\overline{z(\theta)} = \dfrac{1}{z(\theta)} = e^{-i\theta}$. It can be easily calculated that

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z(\theta)+\dfrac{1}{z(\theta)}}{2}, \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z(\theta)-\dfrac{1}{z(\theta)}}{2i}.$$

Plugging the above calculations into (1.152) yields

$$\int_0^{2\pi} F(\sin\theta, \cos\theta)\,\mathrm{d}\theta = \int_0^{2\pi} F\left(\frac{z(\theta) - \dfrac{1}{z(\theta)}}{2i}, \frac{z(\theta) + \dfrac{1}{z(\theta)}}{2}\right)\mathrm{d}\theta.$$

On the other hand we have $z'(\theta) = ie^{i\theta} = iz(\theta)$. The above equality can be rewritten as

$$\int_0^{2\pi} F(\sin\theta, \cos\theta)\,\mathrm{d}\theta = \int_0^{2\pi} F\left(\frac{z(\theta) - \dfrac{1}{z(\theta)}}{2i}, \frac{z(\theta) + \dfrac{1}{z(\theta)}}{2}\right)\frac{z'(\theta)}{iz(\theta)}\,\mathrm{d}\theta.$$

In light of the definition of contour integration, the last equality equivalently gives us

$$\int_0^{2\pi} F(\sin\theta, \cos\theta)\,\mathrm{d}\theta = \int_{\mathrm{Cir}(0;1)} F\left(\frac{z - \dfrac{1}{z}}{2i}, \frac{z + \dfrac{1}{z}}{2}\right)\frac{1}{iz}\,\mathrm{d}z. \tag{1.153}$$

Here $\mathrm{Cir}(0;1)$ is counter-clockwisely oriented. For some typical F, the right-hand side above can be evaluated by residue theorem.

Example 1. By (1.153), it holds

$$\int_0^{2\pi} \frac{1}{1 + a\sin\theta}\,\mathrm{d}\theta = \int_{\mathrm{Cir}(0;1)} \frac{1}{1 + a\dfrac{z - z^{-1}}{2}}\frac{1}{iz}\,\mathrm{d}z = \int_{\mathrm{Cir}(0;1)} \frac{2/a}{z^2 + (2i/a)z - 1}\,\mathrm{d}z. \tag{1.154}$$

Here a is a real number between -1 and 1. The quadratic formula reveals that the denominator of the integrand here has the pure imaginary zeros

$$z_1 = \left(\frac{-1 + \sqrt{1 - a^2}}{a}\right)i, \qquad z_2 = \left(\frac{-1 - \sqrt{1 - a^2}}{a}\right)i.$$

So if $f(z)$ denotes the integrand in the last integral of (1.154), then

$$f(z) = \frac{2/a}{(z - z_1)(z - z_2)}.$$

Notice that because $|a| < 1$,

$$|z_2| = \frac{1 + \sqrt{1 - a^2}}{|a|} > 1.$$

Also, since $|z_1 z_2| = 1$, it follows that $|z_1| < 1$. Hence there are no singular points on $\mathrm{Cir}(0;1)$, and the only singularity of f interior to $\mathrm{Cir}(0;1)$ is the point z_1. By residue theorem, it holds

$$\int_{\mathrm{Cir}(0;1)} \frac{2/a}{z^2 + (2i/a)z - 1}\,\mathrm{d}z = 2\pi i\frac{2/a}{z_1 - z_2} = \frac{2\pi}{\sqrt{1 - a^2}}.$$

Example 2. Now we consider for $a \in (-1, 1)$ the integral

$$\int_0^{\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2}\,\mathrm{d}\theta.$$

By property of cos function and (1.153), it holds

$$\int_0^{\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2}\,\mathrm{d}\theta = \frac{1}{2}\int_0^{2\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2}\,\mathrm{d}\theta = \frac{i}{4}\int_{\mathrm{Cir}(0;1)} \frac{z^4 + 1}{(z - a)(az - 1)z^2}\,\mathrm{d}z.$$

In this case $z = a$ and $z = 0$ are singularities of the function

$$f(z) = \frac{z^4 + 1}{(z-a)(az-1)z^2}$$

in $D(0;1)$. Therefore it holds

$$\int_{\mathrm{Cir}(0;1)} \frac{z^4+1}{(z-a)(az-1)z^2}\,\mathrm{d}z = 2\pi i \left(\mathrm{Res}(f(z);0) + \mathrm{Res}(f(z);a)\right).$$

By (1.117),

$$\mathrm{Res}(f(z);0) = \left[\frac{z^4+1}{(z-a)(az-1)}\right]'(0) = \frac{a^2+1}{a^2}.$$

Moreover

$$\mathrm{Res}(f(z);a) = \left[\frac{z^4+1}{(az-1)z^2}\right](a) = \frac{a^4+1}{(a^2-1)a^2}.$$

Therefore the above arguments all infer that

$$\int_0^\pi \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2}\,\mathrm{d}\theta = -\frac{\pi}{2}\left(\frac{a^2+1}{a^2} + \frac{a^4+1}{(a^2-1)a^2}\right) = \frac{\pi a^2}{1-a^2}.$$

Sect. 20. Argument Principle. In this section we assume l is a smooth simply connected closed curve. Ω is the simply connected region enclosed by l. We call f a meromorphic function in $\overline{\Omega}$ if there are finitely many points P_1, ..., P_N in Ω so that f is analytic on

$$\overline{\Omega} \sim \left\{P_1, ..., P_N\right\}.$$

Moreover P_1, ..., P_N are poles of f with order n_1, ..., n_N, respectively. By Laurent series expansion, we have

$$f(z) = \frac{g_j(z)}{(z-P_j)^{n_j}}, \qquad \text{near each } P_j. \tag{1.155}$$

Here for each $j = 1, ..., N$, g_j is analytic at P_j with $g_j(P_j) \neq 0$. Function f may have finitely many zeros if f is not a constant function. We denote by Z_1, ..., Z_M the M locations of zeros of f. We assume that Z_1, ..., Z_M are not on l. By Taylor expansion of f, we may assume

$$f(z) = (z - Z_j)^{m_j} h_j(z), \qquad \text{near } Z_j. \tag{1.156}$$

Here for each $j = 1, ..., M$, h_j is analytic at Z_j with $h_j(Z_j) \neq 0$. m_j is a finite natural number. By multiple connected version of Cauchy theorem, for $\epsilon > 0$ sufficiently small, it holds

$$\int_l \frac{f'(z)}{f(z)}\,\mathrm{d}z = \sum_{j=1}^M \int_{\mathrm{Cir}(Z_j;\epsilon)} \frac{f'(z)}{f(z)}\,\mathrm{d}z + \sum_{k=1}^N \int_{\mathrm{Cir}(P_k;\epsilon)} \frac{f'(z)}{f(z)}\,\mathrm{d}z.$$

Here $\mathrm{Cir}(z_j;\epsilon)$ and $\mathrm{Cir}(P_k;\epsilon)$ are all counter-clockwisely oriented. Plugging (1.155)-(1.156) to the right-hand side above yields

$$\begin{aligned}
\int_l \frac{f'(z)}{f(z)}\,\mathrm{d}z &= \sum_{j=1}^M \int_{\mathrm{Cir}(Z_j;\epsilon)} \frac{f'(z)}{f(z)}\,\mathrm{d}z + \sum_{k=1}^N \int_{\mathrm{Cir}(P_k;\epsilon)} \frac{f'(z)}{f(z)}\,\mathrm{d}z \\
&= \sum_{j=1}^M \int_{\mathrm{Cir}(Z_j;\epsilon)} \frac{m_j(z-Z_j)^{m_j-1}h_j(z) + (z-Z_j)^{m_j}h_j'(z)}{(z-Z_j)^{m_j}h_j(z)}\,\mathrm{d}z \\
&+ \sum_{k=1}^N \int_{\mathrm{Cir}(P_k;\epsilon)} \frac{-n_k(z-P_k)^{-n_k-1}g_k(z) + (z-P_k)^{-n_k}g_k'(z)}{(z-P_k)^{-n_k}g_k(z)}\,\mathrm{d}z
\end{aligned}$$

$$= \sum_{j=1}^M m_j \int_{\mathrm{Cir}(Z_j;\epsilon)} \frac{1}{z-Z_j}\,\mathrm{d}z - \sum_{k=1}^N n_k \int_{\mathrm{Cir}(P_k;\epsilon)} \frac{1}{z-P_k}\,\mathrm{d}z + \sum_{j=1}^M \int_{\mathrm{Cir}(Z_j;\epsilon)} \frac{h_j'(z)}{h_j(z)}\,\mathrm{d}z + \sum_{k=1}^N \int_{\mathrm{Cir}(P_k;\epsilon)} \frac{g_k'(z)}{g_k(z)}\,\mathrm{d}z.$$

It holds that

$$\int_{\mathrm{Cir}(Z_j;\epsilon)} \frac{h_j'(z)}{h_j(z)}\,\mathrm{d}z = \int_{\mathrm{Cir}(P_k;\epsilon)} \frac{g_k'(z)}{g_k(z)}\,\mathrm{d}z = 0,$$

since h_j and g_k are analytic near Z_j and P_k, respectively with $h_j(Z_j) \neq 0$, $g_k(P_k) \neq 0$. Therefore the last two equalities yield

$$\int_l \frac{f'(z)}{f(z)}\,\mathrm{d}z = 2\pi i \left(\sum_{j=1}^{M} m_j - \sum_{k=1}^{N} n_k \right).$$

Equivalently it follows

$$\frac{1}{2\pi i}\int_l \frac{f'(z)}{f(z)}\,\mathrm{d}z = Z - P, \tag{1.157}$$

where $Z = \sum_{j=1}^{M} m_j$ is the number of zeros counting multiplicities. $P = \sum_{k=1}^{N} n_k$ is the number of poles counting multiplicities.

We can use a different way to calculate

$$\int_l \frac{f'(z)}{f(z)}\,\mathrm{d}z.$$

Let $z(t)$ with $t \in [a, b]$ be a parametrization of l, which induces counter-clockwise direction on l. Moreover

$$z(a) = z(b). \tag{1.158}$$

By definition of contour integral, it holds

$$\int_l \frac{f'(z)}{f(z)}\,\mathrm{d}z = \int_a^b \frac{f'(z(t))}{f(z(t))} z'(t)\,\mathrm{d}t = \int_a^b \frac{\frac{\mathrm{d}}{\mathrm{d}t} f(z(t))}{f(z(t))}\,\mathrm{d}t. \tag{1.159}$$

The last equality uses chain rule. Let $\Gamma(t) = f(z(t))$ with $t \in [a, b]$. Then $\Gamma(t)$ is the parametrization of the image of l under the mapping $f(z)$. (1.159) can then be reduced to

$$\int_l \frac{f'(z)}{f(z)}\,\mathrm{d}z = \int_a^b \frac{\Gamma'(t)}{\Gamma(t)}\,\mathrm{d}t.$$

By the definition of contour integral, it holds

$$\int_a^b \frac{\Gamma'(t)}{\Gamma(t)}\,\mathrm{d}t = \int_\Gamma \frac{1}{w}\,\mathrm{d}w, \qquad \text{where } \Gamma \text{ is the image of } l \text{ under the mapping } f(z).$$

Then the last two equalities yield

$$\int_l \frac{f'(z)}{f(z)}\,\mathrm{d}z = \int_\Gamma \frac{1}{w}\,\mathrm{d}w, \qquad \text{where } \Gamma \text{ is the image of } l \text{ under the mapping } f(z). \tag{1.160}$$

By (1.158), Γ is also a smooth closed curve in $\mathbb{C}$. Since f does not vanish on l, the curve Γ does not pass the origin. Now we want to represent $\Gamma(t)$ by polar coordinates as follows:

$$\Gamma(t) = f(z(t)) = \rho(t)\, e^{i\,\Theta(t)}, \qquad t \in [a, b]. \tag{1.161}$$

Firstly $\rho(t)$ must be $|f(z(t))|$ for all $t \in [a, b]$. Therefore it is a smooth function with respect to the variable $t \in [a, b]$. By (1.158), it also satisfies

$$\rho(a) = |f(z(a))| = |f(z(b))| = \rho(b). \tag{1.162}$$

Since argument function is multiple-valued, to decide the function $\Theta(t)$ is tricky. Notice that Γ is a smooth curve in $\mathbb{C}$ without passing across the origin 0. We can separate the parameter space $[a, b]$ into finitely many small sub-intervals, denoted by $I_j := [t_{j-1}, t_j]$, $j = 1, ..., K$. The union of these sub-intervals equal to $[a, b]$. Moreover it holds $t_0 = a$ and $t_K = b$. We can choose

$$\max\Big\{ |t_j - t_{j-1}| : j = 1, ..., K \Big\}$$

to be small so that on each I_j, the image of $\Gamma(t)$ lies in a branch of a log-function. For points of $\Gamma(t)$ with $t \in I_1$, we can firstly fix a log-function so that the image of $\Gamma(t)$ with $t \in I_1$ are contained in the branch of this log-function. This log-function is denoted by $\log_{[1]}$. Now $\log_{[1]} z$ is analytic at all points on $\Gamma(t)$ with $t \in I_1$. Therefore we can define

$$\Theta_1(t) = \frac{\log_{[1]} \Gamma(t) - \ln \rho(t)}{i}, \qquad \text{for } t \in I_1.$$

The value of $\Theta_1(t)$ with $t \in I_1$ lies in the branch of $\log_{[1]}$. Clearly this $\Theta_1(t)$ is smooth on I_1. Moreover by the last equality it holds

$$\Gamma(t) = \rho(t) e^{i\Theta_1(t)}, \qquad t \in I_1.$$

Now we consider points on I_2. As before we can also find a log-function, denoted by log, so that the image of $\Gamma(t)$ with $t \in I_2$ are contained in the branch of log. But notice that at $\Gamma(t_1)$, the argument of $\Gamma(t_1)$ in the branch of $\log_{[1]}$ differs from the argument of $\Gamma(t_1)$ in the branch of log by $2k\pi$. That is

$$\log_{[1]} \Gamma(t_1) - \log \Gamma(t_1) = 2k\pi i.$$

Therefore by letting $\log_{[2]} z = \log z + 2k\pi i$, we not only can have the analyticity of $\log_{[2]}$ on points of $\Gamma(t)$ with $t \in I_2$. Also we can have

$$\log_{[2]} \Gamma(t_1) = \log_{[1]} \Gamma(t_1). \tag{1.163}$$

Now we similarly define

$$\Theta_2(t) = \frac{\log_{[2]} \Gamma(t) - \ln \rho(t)}{i}, \qquad \text{for } t \in I_2.$$

Clearly this $\Theta_2(t)$ is smooth on I_2. By (1.163), it also holds

$$\Theta_2(t_1) = \Theta_1(t_1).$$

Inductively we can find a sequence of smooth angular functions, denoted by $\Theta_j(t)$, on I_j. Here $j = 1, ..., K$. Moreover for each $j = 1, ..., K-1$, Θ_j satisfies $\Theta_j(t_j) = \Theta_{j+1}(t_j)$. In terms of these Θ_j, we define for all $t \in [a, b]$ an angular function $\Theta(t)$ by

$$\Theta(t) = \Theta_j(t), \qquad \text{if } t \in I_j.$$

Clearly this Θ is a continuous function on $[a, b]$. Moreover Θ is piece-wisely differentiable and satisfies (1.161). The above arguments give us a way to continuously change argument from $\Gamma(a)$ to $\Gamma(b)$ along the curve Γ. By (1.158) and (1.161)-(1.162), we have

$$e^{i\Theta(a)} = e^{i\Theta(b)}.$$

Due to periodicity of sin and cos functions, $\Theta(a)$ and $\Theta(b)$ may not equal to each other. Now we denote by $\Delta_l \arg f(z)$ the difference $\Theta(b) - \Theta(a)$. Clearly this difference must be $2k\pi$ for some integer k. Summarizing the above arguments gives us

$$\Theta(b) - \Theta(a) = \Delta_l \arg f(z) = 2k\pi. \tag{1.164}$$

By (1.161), it satisfies

$$\int_\Gamma \frac{1}{w}\,\mathrm{d}w = \int_a^b \frac{\rho'(t)\,e^{i\,\Theta(t)} + i\,\Theta'(t)\,\rho(t)\,e^{i\,\Theta(t)}}{\rho(t)\,e^{i\,\Theta(t)}}\,\mathrm{d}t = \int_a^b \frac{\rho'(t)}{\rho(t)}\,\mathrm{d}t + i\int_a^b \Theta'(t)\,\mathrm{d}t.$$

By fundamental theorem of calculus, it holds

$$\int_a^b \frac{\rho'(t)}{\rho(t)}\,\mathrm{d}t = \ln\rho(b) - \ln\rho(a), \qquad \int_a^b \Theta'(t)\,\mathrm{d}t = \sum_{j=1}^{K}\int_{t_{j-1}}^{t_j} \Theta'(t)\,\mathrm{d}t = \sum_{j=1}^{K}\Big[\Theta(t_j) - \Theta(t_{j-1})\Big] = \Theta(b) - \Theta(a).$$

Hence (1.162) and (1.164) imply that

$$\int_\Gamma \frac{1}{w}\,\mathrm{d}w = i\big[\Theta(b) - \Theta(a)\big] = i\,\Delta_l\,\mathrm{arg}f(z). \tag{1.165}$$

Combining this result with (1.160), we have

$$\int_l \frac{f'(z)}{f(z)}\,\mathrm{d}z = i\,\Delta_l\,\mathrm{arg}f(z).$$

In light of this equality and (1.157), we get the so-called argument principle. That is

Theorem 1.52 (Argument Principle)**.** *Let l denote a counter-clockwisely oriented smooth closed contour, and suppose that*

(a). a function $f(z)$ is meromorphic in the domain enclosed by l;

(b). $f(z)$ is analytic and non-zero on l;

(c). counting multiplicities, Z is the number of zeros and P is the number of poles of $f(z)$ inside l.

Then

$$\frac{1}{2\pi}\Delta_l\,\mathrm{arg}f(z) = Z - P. \tag{1.166}$$

Example 1. The only zeros of the function

$$f(z) = \frac{z^3+2}{z}$$

are exterior to the circle $|z| = 1$, since they are the cubic roots of -2; and the only singularity in the finite plane is a simple pole at the origin. Hence, if l denotes the circle $|z| = 1$ in the counter-clockwisely orientation, (1.166) tells us that

$$\Delta_l\,\mathrm{arg}f(z) = 2\pi(0-1) = -2\pi.$$

That is, Γ, the image of l under the transformation $f(z)$, winds around the origin once in the clockwise direction.

Sect. 21. Counting Zeros. In this section we apply argument principle introduced in Sect. 20 to count number of zeros for analytic functions. Firstly we give Rouché's theorem, which is a useful criterion to compare number of zeros between two analytic functions.

Theorem 1.53 (Rouché's theorem)**.** *Let l denote a simple closed contour, and suppose that*

(a). two functions $f(z)$ and $g(z)$ are analytic in the domain enclosed by l;

(b). $f(z)$ and $g(z)$ are also analytic on l;

(c). $|f(z)| > |g(z)|$ at each point on l.

Then $f(z)$ and $f(z)+g(z)$ have the same number of zeros, counting multiplicities, on the region enclosed by l.

Proof. By (c) in the hypothesis, we have $|f(z)| > 0$ on l. Therefore

$$f(z) + g(z) = f(z)\left(\frac{g(z)}{f(z)} + 1\right), \qquad z \in l \tag{1.167}$$

is well-defined for all points on l. Letting $z(t)$ with $t \in [a, b]$ be a parametrization of l which is counter-clockwisely oriented, similarly to the arguments in Sect. 20, we can find

$$(\rho_1(t), \Theta_1(t)), \qquad (\rho_2(t), \Theta_2(t)) \qquad \text{and} \qquad (\rho_3(t), \Theta_3(t)), \qquad t \in [a, b]$$

so that

$$f(z(t)) = \rho_1(t)\, e^{i\Theta_1(t)}, \qquad f(z(t)) + g(z(t)) = \rho_2(t)\, e^{i\Theta_2(t)}, \qquad \frac{g(z(t))}{f(z(t))} + 1 = \rho_3(t)\, e^{i\Theta_3(t)}.$$

Plugging the three equalities above into (1.167) yields

$$\rho_2(t)\, e^{i\Theta_2(t)} = \rho_1(t)\, \rho_3(t)\, e^{i\Theta_1(t)+i\Theta_3(t)}, \qquad t \in [a, b].$$

Here ρ_j with $j = 1, 2, 3$ are smooth functions. Θ_j with $j = 1, 2, 3$ are continuous and piecewisely differentiable functions. When t runs from a to b, the argument of the left-hand side above changes from $\Theta_2(a)$ to $\Theta_2(b)$. Therefore the total change of argument equals to $\Theta_2(b) - \Theta_2(a)$. On the other hand, the argument of the right-hand side above changes from $\Theta_1(a) + \Theta_3(a)$ to $\Theta_1(b) + \Theta_3(b)$. The total change of argument equals also to $\Theta_1(b) - \Theta_1(a) + \Theta_3(b) - \Theta_3(a)$. Hence we get

$$\Delta_l \arg(f + g) = \Delta_l \arg f + \Delta_l \arg\left(\frac{g}{f} + 1\right). \tag{1.168}$$

Still by (c) in the hypothesis, it holds

$$\left|\left(\frac{g(z)}{f(z)} + 1\right) - 1\right| < 1, \qquad z \in l.$$

Therefore the image of l under the mapping $\dfrac{g(z)}{f(z)} + 1$ is contained in the open disk $D(1; 1)$. The disk $D(1; 1)$ is strictly on the right-half plane. So the image of l under the mapping $\dfrac{g(z)}{f(z)} + 1$ cannot wind around the origin 0. This shows that

$$\Delta_l \arg\left(\frac{g}{f} + 1\right) = 0.$$

Plugging this result into (1.168) yields

$$\Delta_l \arg(f + g) = \Delta_l \arg f.$$

The proof then follows by argument principle since now $f + g$ and f have no poles. □

Example 1. In order to determine the number of roos, counting multiplicities, of the equation $z^4 + 3z^3 + 6 = 0$ inside the circle $\mathrm{Cir}(0; 2)$, write

$$f(z) = 3z^3 \qquad \text{and} \qquad g(z) = z^4 + 6.$$

Then observe that when $|z| = 2$,

$$|f(z)| = 3|z|^3 = 24 \qquad \text{and} \qquad |g(z)| \leqslant |z|^4 + 6 = 22.$$

The conditions in Rouché theorem are thus satisfied. Consequently, since $f(z)$ has three zeros, counting multiplicities, inside $\mathrm{Cir}(0;2)$, so does $f(z)+g(z)$. That is $z^4+3z^3+6=0$ has three roots, counting multiplicities, inside the circle $\mathrm{Cir}(0;2)$.

Example 2. Fundamental theorem of algebra. Suppose that $P(z) = a_0 + a_1 z + ... + a_n z^n$ ($a_n \neq 0$) is a polynomial of degree n ($n \geqslant 1$). Let $f(z) = a_n z^n$ and $g(z) = P(z) - a_n z^n$. Since g is of degree at most $n-1$, it holds

$$|f(z)| > |g(z)|, \qquad \text{for all } z \in \mathrm{Cir}(0;R), \text{ provided that } R \text{ is large enough.}$$

By Rouché theorem, number of zeros of $P(z) = f(z)+g(z)$ equals to the number of zeros of $f(z) = a_n z^n$ in $D(0;R)$, provided that R is large enough. Therefore $P(z)$ has n roots in $\mathbb{C}$.

Example 3. In this last example we consider how many roots of the equation $z^4+8z^3+3z^2+2z+2=0$ lie in the right-half plane. Firstly we check if there are roots on the pure imaginary line. To do so we assume $z = it$ with $t \in \mathbb{R}$ and plug $z = it$ into $P(z) = z^4+8z^3+3z^2+2z+2$. By this way we have

$$P(it) = (t^4 - 3t^2 + 2) + i(-8t^3 + 2t), \qquad t \in \mathbb{R}. \tag{1.169}$$

If $P(it) = 0$ for some $t \in \mathbb{R}$, then

$$t^4 - 3t^2 + 2 = (t^2-1)(t^2-2) = 0. \tag{1.170}$$

Meanwhile

$$2t - 8t^3 = 0. \tag{1.171}$$

But (1.170)-(1.171) have no common roots. So $P(z)$ has no root on pure imaginary line.

Let C_R be the right-half part of $\mathrm{Cir}(0;R)$. We can assume C_R is counter-clockwisely oriented. When R is large enough, there is no root of $P(z)$ on C_R since by fundamental theorem of algebra, we have only 4 roots for the polynomial $P(z)$. Now we denote by l_R the contour constructed as follows. Firstly we go from $-iR$ to iR along C_R. Then we go from iR to $-iR$ downwardly along the pure imaginary line. Clearly the previous arguments imply that there is no root of $P(z)$ on l_R. To count the number of roots of $P(z)$ on the right-half plane, we just need to compute the total change of argument of the image of l_R under the mapping $P(z)$.

I. Change of argument on C_R. For C_R, we parameterize it by $z(\theta) = Re^{i\theta}$, where θ runs from $-\pi/2$ to $\pi/2$. Then we get

$$\int_{C_R} \frac{P'(z)}{P(z)}\,\mathrm{d}z = \int_{-\pi/2}^{\pi/2} \frac{P'(Re^{i\theta})}{P(Re^{i\theta})} Re^{i\theta} i\,\mathrm{d}\theta = \int_{-\pi/2}^{\pi/2} \frac{4R^3e^{i3\theta} + 24R^2e^{i2\theta} + 6Re^{i\theta} + 2}{R^4e^{i4\theta} + 8R^3e^{i3\theta} + 3R^2e^{i2\theta} + 2Re^{i\theta} + 2} Re^{i\theta} i\,\mathrm{d}\theta. \tag{1.172}$$

Since the integrand of the last integral above satisfies

$$\frac{4R^3e^{i3\theta} + 24R^2e^{i2\theta} + 6Re^{i\theta} + 2}{R^4e^{i4\theta} + 8R^3e^{i3\theta} + 3R^2e^{i2\theta} + 2Re^{i\theta} + 2} Re^{i\theta} i \longrightarrow 4i, \qquad \text{as } R \to \infty, \text{ uniformly for all } \theta \text{ on } [-\pi/2, \pi/2].$$

Then taking $R \to \infty$ in (1.172) yields

$$\lim_{R\to\infty} \int_{C_R} \frac{P'(z)}{P(z)}\,\mathrm{d}z = 4\pi i. \tag{1.173}$$

II. Change of argument on pure imaginary line. For the part of l_R on the pure imaginary line, we parameterize it by $z(t) = it$ with t running from R to $-R$. In light of (1.169), the imaginary part of (1.169) equals to zero if and only if $t = 0$, $t = 1/2$ or $t = -1/2$. At these three values for t, the associated values of the real part of (1.169) equal to 2, 21/16 and 21/16, respectively. In other words for $t \in [-R, R]$, $P(it)$ computed in (1.169) can not take values on $\left\{x : x \leqslant 0\right\}$. Therefore the image of $\left\{it : t \in [-R, R]\right\}$ under the mapping $P(z)$ is contained in the branch of principal log-function. It holds

$$\text{Arg}\left(P(iR)\right) = \arctan\left(\frac{2R - 8R^3}{R^4 - 3R^2 + 2}\right), \qquad \text{Arg}\left(P(-iR)\right) = \arctan\left(\frac{-2R + 8R^3}{R^4 - 3R^2 + 2}\right).$$

Therefore along the image of $[iR, -iR]$ under the mapping $P(z)$, the argument changes from

$$\arctan\left(\frac{2R - 8R^3}{R^4 - 3R^2 + 2}\right)$$

to

$$\arctan\left(\frac{-2R + 8R^3}{R^4 - 3R^2 + 2}\right).$$

Here $[iR, -iR]$ is the directional line on the pure imaginary line starting from iR to $-iR$. The total change of argument equals to

$$\Delta_{[iR,-iR]}\,\text{arg}P(z) = 2\arctan\left(\frac{-2R + 8R^3}{R^4 - 3R^2 + 2}\right) \longrightarrow 0, \qquad \text{as } R \to \infty. \tag{1.174}$$

By (1.173), the total change of argument along the image of C_R under $P(z)$ almost equals to 4π. By (1.174), the total change of argument along the image of $[iR, -iR]$ under $P(z)$ almost equals to 0. Therefore the total change of argument along the image of l_R under $P(z)$ must be 4π, when R is large enough. By argument principle it follows $Z - P = 2$. But $P(z)$ has no poles, hence $Z = 2$. That is $P(z)$ totally has 2 roots on the right-half plane.